Completely Bounded Maps and Operator Algebras
Chapter 9: Completely Bounded Homomorphisms
Solutions to Exercises

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9.7 (Sz.-Nagy) This exercise gives a more direct proof of Corollary 9.4. Let T be an invertible operator on $\mathcal{H}$ such that $\|T^n\| \leq M$ for all integers n, and let glim be a Banach generalized limit [34].

(i) Show that $\langle x, y\rangle_1 = \operatorname{glim} \langle T^n x, T^n y\rangle$ defines a new inner product on $\mathcal{H}$ and that $M^{-2} \langle x, x\rangle \leq \langle x, x\rangle_1 \leq M^2 \langle x, x\rangle$.

(ii) Show that T is a unitary transformation on $(\mathcal{H}, \langle , \rangle_1)$.

(iii) Prove that there exists a similarity on $\mathcal{H}$, with $\|S^{-1}\|\|S\| \leq M^2$, such that $S^{-1}TS$ is unitary.

Solution: We begin by noting that glim : $\ell^\infty \to \mathbb{C}$ is a positive linear functional, $\operatorname{glim}((\alpha_n)_{n=1}^\infty) = \lim_{n\to\infty} \alpha_n$ whenever $(\alpha_n)_{n=1}^\infty$ is convergent and $\operatorname{glim}((\alpha_n)_{n=1}^\infty) = \operatorname{glim}((\alpha_{n+1})_{n=1}^\infty)$ for any $(\alpha_n)_{n=1}^\infty \in \ell^\infty$.

(i) We have by the Cauchy-Schwarz inequality that,

$$\langle T^n x, T^n x\rangle \leq \|T^n\|^2 \langle x, x\rangle \leq M^2 \langle x, x\rangle , \tag{1}$$

and,

$$\langle x, x\rangle = \left\langle T^{-n}T^n x, T^{-n}T^n x\right\rangle \leq \left\|T^{-n}\right\|^2 \langle T^n x, T^n x\rangle \leq M^2 \langle T^n x, T^n x\rangle . \tag{2}$$

This shows that $\langle T^n x, T^n x\rangle$ is an ℓ^∞ sequence and so $\langle , \rangle_1$ is well-defined. Since glim is positive and linear $\langle , \rangle_1$ is a semi-inner product on $\mathcal{H}$. Applying glim to the inequalities in (1) and (2) establishes $M^{-2} \langle x, x\rangle \leq \langle x, x\rangle_1 \leq M^2 \langle x, x\rangle$, and so $\langle , \rangle_1$ is an inner product.

(ii) As T is invertible, our claim is established if we prove that T is an isometry. We have,

$$\|Tx\|_1^2 = \langle Tx, Tx\rangle_1 = \operatorname{glim} \left\langle T^{n+1}x, T^{n+1}x\right\rangle = \operatorname{glim} \langle T^n x, T^n x\rangle = \langle x, x\rangle_1 = \|x\|_1^2 . \tag{3}$$

(iii) Using the fact that $M^{-1} \|x\| \leq \|x\|_1 \leq M \|x\|$ we see that $R : (\mathcal{H}, \langle , \rangle) \to (\mathcal{H}, \langle , \rangle_1)$ defined by $R(h) = h$ is a bounded invertible operator with $\|R\| \leq M$ and $\left\|R^{-1}\right\| \leq M$. Therefore the Hilbert spaces $(\mathcal{H}, \langle , \rangle)$ and $(\mathcal{H}, \langle , \rangle_1)$ have the same dimension. It follows that there is a unitary $U : (\mathcal{H}, \langle , \rangle) \to (\mathcal{H}, \langle , \rangle_1)$. The operator,

$$U^* RTR^{-1}U = (R^{-1}U)^{-1}TR^{-1}U, \tag{4}$$

is unitary and similar to T, via the similarity $S = R^{-1}U$. Since U and U^* are unitary we have $\|S\| = \left\|R^{-1}\right\| \leq M$ and $\left\|S^{-1}\right\| = \|R\| \leq M$. Hence $\|S\|\|S^{-1}\| \leq M^2$.

9.10 (Sz.-Nagy-Foias) An operator T in $B(\mathcal{H})$ is said to belong to class $\mathcal{C}_\rho$ if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, and a unitary on $\mathcal{K}$, such that

$$T^n = \rho P_{\mathcal{H}} U^n|_{\mathcal{H}},$$

for all positive integers n. Prove that such a T is completely polynomially bounded and that there exists an invertible operator S such that $S^{-1}TS$ is a contraction with $\|S^{-1}\|\|S\| \leq 2\rho - 1$ when $\rho \geq 1$.

Solution: By Theorem 9.8 it is enough to prove that $\varphi : \mathcal{P}(\mathbb{D}) \to B(\mathcal{H})$ defined by $\varphi(p) = p(T)$ is completely bounded with $\|\varphi\|_{cb} \leq 2\rho - 1$. Let $p(z) = \alpha_0 + \alpha_1 z + \ldots + \alpha_m z^m$. We have,

$$\begin{aligned} p(T) &= \alpha_0 I + \alpha_1 T + \ldots + \alpha_m T^m && (5)\\ &= \alpha_0 I + \rho P_{\mathcal{H}}(\alpha_1 U + \ldots + \alpha_m U^m)|_{\mathcal{H}} && (6)\\ &= \alpha_0 I - \rho\alpha_0 I + \rho\alpha_0 I + \rho P_{\mathcal{H}}(\alpha_1 U + \ldots + \alpha_m U^m)|_{\mathcal{H}} && (7)\\ &= \alpha_0(1-\rho) I + \rho P_{\mathcal{H}} p(U)|_{\mathcal{H}}. && (8) \end{aligned}$$

It follows by the spectral mapping theorem since $p(U)$ is normal that,

$$\|p(T)\| \leq (\rho - 1)\,|p(0)| + \rho\,\|p(U)\| \leq (\rho-1)\,\|p\|_\infty + \rho\,\|p\|_\infty = (2\rho - 1)\,\|p\|_\infty\,. \tag{9}$$

Let $(p_{i,j}) \in M_n(\mathcal{P}(\mathbb{D}))$. We have,

$$\|\varphi_n((p_{i,j}))\| = \|(p_{i,j}(T))\| \leq (\rho-1)\,\|(p_{i,j}(0))\| + \rho\,\|(p_{i,j}(U))\|\,. \tag{10}$$

Since U is a unitary we can apply von Neumann's inequality for matrices (Corollary 3.12),

$$\|(p_{i,j}(U))\|_{B(\mathcal{K}^{(n)})} \leq \|(p_{i,j})\|_{M_n(C(\mathbb{T}))}\,, \tag{11}$$

to (10) to get,

$$\|\varphi_n\| \leq 2\rho - 1. \tag{12}$$

9.12 let T be an operator with real spectrum. Prove that T is similar to a self-adjoint operator if and only if the Cayley transform of T, $C = (T+i)(T-i)^{-1}$, has the property that for some constant M, $\|C^n\| \leq M$ for all integers n.

Solution: Assume first that $\|C^n\| \leq M$ for all integers n. By Sz.-Nagy's theorem $U = SCS^{-1}$ is unitary for some invertible operator S. We claim that STS^{-1} is self-adjoint. Note that,

$$\sigma(C) = \sigma(U) = \left\{ \frac{t+i}{t-i} \,:\, t \in \sigma(T) \right\}, \tag{13}$$

and that $1 \notin \sigma(U)$. From $U = SCS^{-1} = S(T+i)(T-i)^{-1}S^{-1}$ it follows that $US(T-i)S^{-1} = S(T+i)S^{-1}$. On rearranging this equation we get $(U - \mathbf{1})STS^{-1} = i(U + \mathbf{1})$. As $(U - \mathbf{1})$ is invertible $STS^{-1} = i(U-\mathbf{1})^{-1}(U+\mathbf{1})$. Therefore,

$$\begin{aligned} (STS^{-1})^* &= -i(U+\mathbf{1})^*((U-\mathbf{1})^*)^{-1} && (14)\\ &= -i(U^{-1}+\mathbf{1})(U^{-1}-\mathbf{1})^{-1} && (15)\\ &= -i(\mathbf{1}+U)U^{-1}((\mathbf{1}-U)U^{-1})^{-1} && (16)\\ &= -i(\mathbf{1}+U)U^{-1}U(\mathbf{1}-U)^{-1} && (17)\\ &= i(U-\mathbf{1})^{-1}(U+\mathbf{1}) && (18)\\ &= STS^{-1}. && (19) \end{aligned}$$

To prove the converse suppose that $T = SRS^{-1}$ where R is self-adjoint and let B denote the Cayley transform of R. The adjoint of B is given by,

$$\begin{aligned} B^* &= ((R-i)^*)^{-1}(R+i)^* && (20)\\ &= (R+i)^{-1}(R-i) && (21)\\ &= ((R-i)^{-1}(R+i))^{-1} && (22)\\ &= ((R+i)(R-i)^{-1})^{-1} && (23)\\ &= B^{-1}, && (24) \end{aligned}$$

and so B is unitary. Further,

$$\begin{aligned} B &= (S^{-1}TS+i)(S^{-1}TS-i)^{-1} && (25)\\ &= S^{-1}(T+i)S[S^{-1}(T-i)S]^{-1} && (26)\\ &= S^{-1}(T+i)(T-i)^{-1}S && (27)\\ &= S^{-1}CS. && (28) \end{aligned}$$

Hence,

$$\|C^n\| = \left\|S^{-1}B^nS\right\| \leq \left\|S^{-1}\right\| \|B^n\| \|S\| = \|S^{-1}\|\|S\|. \tag{29}$$

9.15 Let T be in $B(\mathcal{H})$ with $\sigma(T)$ contained in the open unit disk. Prove that $P = \sum_{k=0}^{\infty} T^{*k}T^k$ is a norm convergent series with $\left\|P^{1/2}TP^{-1/2}\right\| \leq 1$.

Solution: Let $R > 0$ be chosen so that $r(T) < R < 1$. By the spectral radius formula there exists $N \in \mathbb{N}$ such that $\|T^n\|^{1/n} \leq R$ for all $n \geq N$. Therefore $\|T^n\| = \|T^{*n}\| \leq R^n$ for all $n \geq N$ and so $\|T^{*n}T^n\| \leq R^{2n}$ for all $n \geq N$. It follows that the series is absolutely convergent in $B(\mathcal{H})$ and consequently norm convergent. Since P is the sum of positive operators and $P \geq I$ we see that P is positive and invertible.

Note that,

$$\begin{aligned} \|P^{1/2}TP^{-1/2}\| \leq 1 \quad &\iff \quad (P^{1/2}TP^{-1/2})^*(P^{1/2}TP^{-1/2}) \leq I && (30)\\ &\iff \quad P^{-1/2}T^*PTP^{-1/2} \leq I && (31)\\ &\iff \quad T^*PT \leq P. && (32) \end{aligned}$$

We establish this last inequality,

$$T^*PT = \sum_{k=0}^{\infty} T^*(T^{*k}T^k)T = \sum_{k=0}^{\infty} T^{*k+1}T^{k+1} = \sum_{k=1}^{\infty} T^{*k}T^k = P - I \leq P. \tag{33}$$