Completely Bounded Maps and Operator Algebras
Chapter 10: Polynomially Bounded Operators
Solutions to Exercises

January 24, 2006

10.1 Let

$$J_\lambda = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & \ddots & 0 \\ \vdots & \ddots & & \ddots & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{bmatrix} \in M_k$$

be an elementary Jordan block.

(i) Show that the $(1,k)$ entry of J_λ^m is the $(k-1)$-th derivative of $z^m/(k-1)!$ evaluated at λ. Deduce that J_λ is power bounded if and only if $|\lambda| < 1$.

(ii) Prove that if $|\lambda| < 1$ then J_λ is similar to a contraction.

(iii) Let $T \in M_n$ be power bounded. Prove that T is similar to a contraction.

Solution:

(i) We write $J_\lambda = \lambda I + N$ and note that N is nilpotent of order k. Therefore,

$$\begin{aligned} J_\lambda^m &= (\lambda I + N)^m && (1) \\ &= \sum_{j=0}^{\min\{k-1,m\}} \binom{m}{j} \lambda^{m-j} N^j && (2) \\ & && (3) \end{aligned}$$

If $m \leq k-1$ then the $(1,k)$ entry is 0 which is also the $(k-1)$th derivative of $z^m/(k-1)!$. For $m \geq k$ The $(1,k)$ entry is given by,

$$\binom{m}{k-1} \lambda^{m-k+1} = \frac{m(m-1)\dots(m-k+2)}{(k-1)!} \lambda^{m-k+1} = \left. \frac{d^{k-1}}{dz^{k-1}} \right|_{z=\lambda} \frac{z^m}{(k-1)!}. \tag{4}$$

If $\lambda \geq 1$ then

$$\|J_\lambda^m\| \geq \binom{m}{k-1} |\lambda|^{m-k+1} \to \infty \text{ as } m \to \infty. \tag{5}$$

Therefore, J_λ is not power bounded. Conversely if $|\lambda| < 1$, then for $m \geq k-1$

$$\|J_\lambda^m\| \leq \sum_{j=0}^{k-1} \binom{m}{j-1} |\lambda|^{m-j} \leq k m^k |\lambda|^{m-j} \to 0 \text{ as } m \to \infty. \tag{6}$$

(ii) Let

$$D_r = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & r & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r^{k-1}. \end{bmatrix}$$

If $r \neq 0$, then

$$D_r^{-1} J_\lambda D_r = \begin{bmatrix} \lambda & r & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & \ddots & 0 \\ \vdots & \ddots & & \ddots & r \\ 0 & \cdots & 0 & 0 & \lambda \end{bmatrix} \tag{7}$$

and so

$$\|D_{r^{-1}} J_\lambda D_r\| \leq |\lambda| + r \leq 1, \tag{8}$$

for any $0 < r \leq 1 - |\lambda|$.

(iii) As T is power bounded there exists a constant M such that $\|T^n\| \leq M$ for all $n \geq 0$. Let J denote the Jordan form of T and suppose that $J = \operatorname{diag}(J_{\lambda_1}, \dots, J_{\lambda_l})$, where J_{λ_i} is an elementary Jordan block. Suppose that $R^{-1}TR = J$. We have that,

$$J^n = R^{-1}T^nR = \operatorname{diag}(J_{\lambda_1}^n, \dots, J_{\lambda_l}^n) \tag{9}$$

and,

$$\left\|J_{\lambda_i}^n\right\| \leq \left\|R^{-1}T^nR\right\| \leq M \left\|R^{-1}\right\| \|R\|. \tag{10}$$

Hence J_{λ_i} is power bounded. It follows by part (i) that $|\lambda_i| < 1$ for $i = 1, \dots, l$ and so by part (ii) there exist similarities $S_1, \dots, S_l$ such that $S_i^{-1} J_{\lambda_i} S_i$ is a contraction. Let $D = \operatorname{diag}(S_1, \dots, S_l)$, $S = RD$ and note that

$$\left\|S^{-1}TS\right\| = \left\|D^{-1}R^{-1}TRD\right\| = \left\|D^{-1}JD\right\| = \max_{i=1,\dots,l} \left\|S_i^{-1} J_{\lambda_i} S_i\right\| \leq 1. \tag{11}$$

10.3 i) Show that g_1, f_1, p_1, q_1 satisfy 1)-4) of Theorem 10.8.

ii) Show that if g_m, f_m, p_m, q_m satisfy 1)-4), then $g_{m+1}, f_{m+1}, p_{m+1}, q_{m+1}$ satisfy 1)-4).

Solution:

i) We have $g_1 \equiv q_1 \equiv 1$, $p_1(z) = -\overline{h_1} z^{k_1}$ which has degree at most k_1 and $f_1(z) = h_1 z^{k_1}$ which implies $\Gamma(f_1) = (h_1, 0, \dots)$.

ii) We compute

$$\begin{aligned} F_{m+1}(e^{i\theta}) &= F_m(e^{i\theta}) B_{m+1}(e^{i\theta}) && (12) \\ &= \begin{bmatrix} g_m(e^{i\theta}) & p_m(e^{-i\theta}) \\ f_m(e^{i\theta}) & q_m(e^{-i\theta}) \end{bmatrix} \begin{bmatrix} 1 & -\overline{h}_{m+1} e^{-ik_{m+1}\theta} \\ h_{m+1} e^{ik_{m+1}\theta} & 1 \end{bmatrix} && (13) \\ &= \begin{bmatrix} g_m(e^{i\theta}) + h_{m+1} e^{ik_{m+1}\theta} p_m(e^{-i\theta}) & p_m(e^{-i\theta}) - \overline{h}_{m+1} e^{-ik_{m+1}\theta} g_m(e^{i\theta}) \\ f_m(e^{i\theta}) + h_{m+1} e^{ik_{m+1}\theta} q_m(e^{-i\theta}) & q_m(e^{-i\theta}) - \overline{h}_{m+1} e^{-ik_{m+1}\theta} f_m(e^{i\theta}) \end{bmatrix} && (14) \end{aligned}$$

We will deduce properties 1) and 4), the remaining deductions are similar. We have,

$$g_{m+1}(e^{i\theta}) = g_m(e^{i\theta}) + h_{m+1}e^{ik_{m+1}\theta}p_m(e^{-i\theta}). \tag{15}$$

By assumption the degree of g_m and p_m is at most k_m but $p_m(e^{i\theta})$ is a polynomial in negative powers of $e^{i\theta}$ and so the degree of the second term in (15) is at most k_{m+1}. The only contribution to the constant term in g_{m+1} is from g_m, since $k_{m+1} > k_m$. It follows that $g_m(0) = 1$. Now consider

$$f_{m+1}(e^{i\theta}) = f_m(e^{i\theta}) + h_{m+1}e^{ik_{m+1}\theta}q_m(e^{-i\theta}). \tag{16}$$

Since $k_{m+1} > 2k_m + 1$ the smallest exponent that appears in $h_{m+1}e^{ik_{m+1}\theta}q_m(e^{-i\theta})$ is at least $k_{m+1} - k_m > k_m + 1$. In addition the constant term of q_m is 1 and so the coefficient of $z^{k_{m+1}}$ is h_{m+1}. Therefore, the coefficients of z^k, $k = 0, \ldots, k_m$, in f_m and f_{m+1} are the same and the coefficient of $z^{k_{m+1}}$ in f_{m+1} is h_{m+1}. Hence, $\Gamma(f_{m+1}) = (h_1, \ldots, h_m, h_{m+1}, 0, \ldots)$.

10.4 Let Γ be the map of Theorem 10.8. Prove that $\dot{\Gamma} : A(\mathbb{D})/\ker\Gamma \to \ell^2$ is a Banach space isomorphism with $\|\dot{\Gamma}\| \leq 1, \|\dot{\Gamma}^{-1}\| \leq \sqrt{e}$.

Solution: By standard results on quotient spaces and using the fact that Γ is contractive and onto we have that $\dot{\Gamma}$ is a bijective, contractive map. From the proof of Theorem 10.8 we see that given $h \in \ell^2$ there exists a $f \in A(\mathbb{D})$ such that $\Gamma(f) = h$ and $\|f\| \leq \sqrt{e}\,\|h\|$. Thus,

$$\|\dot{\Gamma}^{-1}(h)\| = \|f + \ker\Gamma\| \leq \|f\| \leq \sqrt{e}\,\|h\|\,, \tag{17}$$

and so $\|\dot{\Gamma}^{-1}\| \leq \sqrt{e}$

10.5 Let $C_1, \ldots, C_n$ denote the $2^n \times 2^n$ CAR matrices, let $E_{1,1}, \ldots, E_{n,1}$ be the standard matrix units in M_n and let $\Phi(\lambda_1 E_{1,1} + \ldots + \lambda_n E_{n,1}) = \lambda_1 C_1 + \ldots + \lambda_n C_n$, so that Φ is an isometry. Prove that $\|\Phi\|_{cb} \geq \sqrt{n}/2$.

Solution: Let $C = \lambda_1 C_1 + \ldots + \lambda_n C_n$. Let $P = CC^*$ and $Q = C^*C$. Note that

$$C^2 = \sum_{1\leq i<j\leq n} \lambda_i\lambda_j(C_iC_j + C_jC_i) + \sum_{j=1}^{n} \lambda_j^2 C_j^2 = 0. \tag{18}$$

From this we get $QP = C^*C^2C^* = 0$. Moreover,

$$P + Q = CC^* + C^*C = \sum_{i,j=1}^{n} \lambda_i\overline{\lambda_j}(C_iC_j^* + C_i^*C_j) = \sum_{i=1}^{n} |\lambda_i|^2\, I. \tag{19}$$

It follows that $\|P\| = \|Q\| = \|P + Q\| = \sum_{i=1}^{n} |\lambda_i|^2$. Therefore Φ is an isometry.

Recall that $C_iC_i^* = I_2^{\otimes(i-1)} \otimes E_{2,2} \otimes I_2^{\otimes(n-i)}$, $i = 1, \ldots, n$. Therefore $C_iC_i^*$ is a diagonal matrix with ones and zeroes on the diagonal. Inspection reveals that the pattern of ones and zeros is 2^{i-1} zeroes followed by 2^{i-1} ones and so on down the diagonal. Therefore the matrix $\sum_{i=1}^{n} C_iC_i^*$ is diagonal with n being the largest diagonal entry. Hence, $\|\sum_{i=1}^{n} C_iC_i^*\| = n$.

Let $A \in M_n(M_n)$ be the matrix.

$$\begin{bmatrix} E_{1,1} & \ldots & E_{n,1} \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{bmatrix}. \tag{20}$$

By permuting the columns of A we see that A is unitarily equivalent to the matrix which has 1's in the first n diagonal entries and is zero otherwise. Therefore $\|A\| = 1$. Now,

$$\|\Phi_n(A)\Phi_n(A)^*\| = \left\| \begin{bmatrix} C_1 & \dots & C_n \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} C_1^* & \dots & 0 \\ \vdots & & \vdots \\ C_n^* & \dots & 0 \end{bmatrix} \right\| \tag{21}$$

$$= \left\| \sum_{i=1}^{n} C_i C_i^* \right\| = n. \tag{22}$$

Hence $\|\Phi\|_{cb} \geq \|\Phi_n\| \geq \sqrt{n}$.