Completely Bounded Maps and Operator Algebras
Chapter 13: Abstract Characterizations of Operator Systems and Operator Spaces
Solutions to Exercises

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13.1 Let V be a matrix normed space, let $X \in M_{n,m}(V)$ and let $Y \in M_{p,q}(V)$ be a matrix obtained from X by introducing finitely many rows and columns of 0's. Prove that $\|X\|_{m,n} = \|Y\|_{p,q}$.

Solution: Let

$$A = \left[\begin{array}{c|c} I_m & 0 \end{array}\right] \in M_{m,p}, B = \begin{bmatrix} I_n \\ 0 \end{bmatrix} \in M_{q,n}. \tag{1}$$

We see then that $X = AYB$ and so $\|X\|_{m,n} \leq \|A\| \|Y\|_{p,q} \|B\| \leq \|Y\|_{p,q}$. Now setting

$$A = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \in M_{p,m}, B = \left[\begin{array}{c|c} I_n & 0 \end{array}\right] \in M_{n,q}, \tag{2}$$

we see that $AXB = Y$ and so $\|Y\|_{p,q} \leq \|A\| \|X\|_{m,n} \|B\| = \|X\|_{m,n}$. Hence $\|X\|_{m,n} = \|Y\|_{p,q}$.

13.2 Let V be a vector space and assume that we are given a sequence of norms, $\|\cdot\|_n$ on $M_n(V)$ satisfying:

(i) $\|AXB\|_n \leq \|A\| \|X\|_n \|B\|$ for $X \in M_n(V)$, $A \in M_n$ and $B \in M_n$.
(ii) for $X \in M_n(V)$, $\|X \oplus 0\|_{m+n} = \|X\|_n$, where 0 denotes an $m \times m$ matrix of 0's.

For $X \in M_{m,n}(V)$ set $\|X\|_{m+n} = \left\|\widehat{X}\right\|_l$ where $l = \max\{m, n\}$ and $\widehat{X}$ is the matrix obtained by adding sufficiently many rows or columns to X to make it square. Prove that $(V, \|\cdot\|_{m,n})$ is a matrix normed space.

These alternate axioms are often given as the axioms for a matrix normed space and, consequently, no mention is given of the norms of rectangular matrices.

Solution: Let $A \in M_{p,m}$, $X \in M_{m,n}$ and $B \in M_{n,q}$. Let $s = \max\{m, n, p, q\}$ and inflate A, B, X by introducing rows and columns of zeros to make them $s \times s$ matrices. Let $l = \max\{p, q\} \leq s$, $k = \max\{m, n\}$. Now,

$$\begin{aligned}
\|AXB\|_{p,q} &= \left\|\widehat{AXB}\right\|_l && (3)\\
&= \left\|\begin{bmatrix} AXB & 0 \\ 0 & 0 \end{bmatrix}\right\| && (4)\\
&\leq \left\|\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right\| \left\|\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\right\|_s \left\|\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}\right\| && (5)\\
&= \|A\| \|\widehat{X} \oplus 0\|_s \|B\| && (6)\\
&= \|A\| \|\widehat{X}\|_k \|B\| && (7)\\
&= \|A\| \|X\|_{m,n} \|B\|. && (8)
\end{aligned}$$

13.3 Let V be an operator space and let W be a closed subspace and let $\pi : V \to V/W$ denote the quotient map $\pi(v) = v + W$. Prove that if we define norms on $M_{m,n}(V/W)$ by setting

$$\|(\pi(v_{i,j}))\|_{m,n} = \inf\{\|v_{i,j} + w_{i,j}\|_{m,n} : w_{i,j} \in W\}$$

then V/W is an operator space.

Solution: Let $A \in M_{p,m}$ $X \in M_{m,n}$ and $B \in M_{n,q}$. Let $\varepsilon > 0$ and choose Z such that $\|X\|_{m,n} + \varepsilon > \|Z\|_{m,n}$ with $\pi_{m,n}(Z) = X$. It is a straighforward calculation to check that $\pi_{p,q}(AZB) = A\pi_{m,n}(Z)B = AXB$. Therefore,

$$\begin{aligned} \|AXB\|_{p,q} &\leq \|AZB\|_{p,q} && (9)\\ &\leq \|A\| \|Z\|_{m,n} \|B\| && (10)\\ &\leq \|A\| \|X\|_{m,n} \|B\| + \varepsilon \|A\| \|B\| . && (11) \end{aligned}$$

As the choice of ε was arbitrary we see that $\|AXB\|_{p,q} \leq \|A\| \|X\|_{m,n} \|B\|$.

We now check that this is an L^∞-matrix normed space. To do this we need only check that

$$\|X \oplus Y\|_{m+p,n+q} \leq \max\{\|X\|_{m,n}, \|Y\|_{p,q}\}$$

where $X \in M_{m,n}(V/W)$ and $Y \in M_{p,q}(V/W)$. Choose R, Z such that $\pi_{m,n}(R) = X$, $\pi_{p,q}(Z) = Y$ and $\|X\|_{m,n} + \varepsilon > \|Z\|_{m,n}$, $\|Y\|_{p,q} + \varepsilon > \|R\|_{p,q}$. Note that $\pi_{m+p,n+q}(R \oplus Z) = X \oplus Y$. We have,

$$\begin{aligned} \|X \oplus Y\|_{m+p,n+q} &\leq \|R \oplus Z\|_{m+p,n+q} && (12)\\ &= \max\{\|R\|_{m,n}, \|Z\|_{p,q}\} && (13)\\ &\leq \max\{\|X\|_{m,n}, \|Y\|_{p,q}\} + \varepsilon. && (14) \end{aligned}$$

By letting $\varepsilon \to 0$ we get our result.

13.5 Verify the claims of Proposition 13.3.

Solution: Given an a matrix ordered $*$-vector space $\mathcal{S}$ with an Archimedean matrix order unit e we define

$$\|x\| = \inf\left\{ r : \begin{bmatrix} re & x \\ x^* & re \end{bmatrix} \in \mathcal{C}_2 \right\}. \tag{15}$$

Denote by S_x the set on the right hand side of (15). We have already seen that $\|x\| \geq 0$ and that $\|x\| = 0$ if and only if $x = 0$. We will now prove that $\|\lambda x\| = |\lambda| \|x\|$, $\|x + y\| \leq \|x\| + \|y\|$ and $\|x^*\| = \|x\|$. We may assume that $\lambda \neq 0$. Note that,

$$\begin{bmatrix} \lambda^{1/2} & 0 \\ 0 & \overline{\lambda}^{1/2} \end{bmatrix} \begin{bmatrix} re & x \\ x^* & re \end{bmatrix} \begin{bmatrix} \overline{\lambda}^{1/2} & 0 \\ 0 & \lambda^{1/2} \end{bmatrix} = \begin{bmatrix} |\lambda| re & \lambda x \\ \overline{\lambda} x^* & |\lambda| re \end{bmatrix}. \tag{16}$$

It follows that,

$$\begin{bmatrix} re & x \\ x^* & re \end{bmatrix} \in \mathcal{C}_2 \iff \begin{bmatrix} |\lambda| re & \lambda x \\ \overline{\lambda} x^* & |\lambda| re \end{bmatrix} \in \mathcal{C}_2. \tag{17}$$

Hence,

$$\begin{aligned} \|\lambda x\| &= \inf\left\{ r : \begin{bmatrix} re & \lambda x \\ \overline{\lambda} x^* & re \end{bmatrix} \in \mathcal{C}_2 \right\} && (18)\\ &= \inf\left\{ r : \begin{bmatrix} |\lambda|^{-1} re & x \\ x^* & |\lambda|^{-1} re \end{bmatrix} \in \mathcal{C}_2 \right\} && (19)\\ &= \inf\left\{ |\lambda| s : \begin{bmatrix} se & x \\ x^* & se \end{bmatrix} \in \mathcal{C}_2 \right\} && (20)\\ &= |\lambda| \|x\| . && (21) \end{aligned}$$

Let $x, y \in \mathcal{S}$ and note that if $r \in S_x$ and $s \in S_y$, then $r + s \in S_{x+y}$. It follows that, $\inf S_{x+y} \leq r + s$ for every $r \in S_x$ and $s \in S_y$. Therefore $\|x + y\| = \inf S_{x+y} \leq \inf S_x + \inf S_y = \|x\| + \|y\|$.

Finally we see that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} re & x \\ x^* & re \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} re & x^* \\ x & re \end{bmatrix}. \tag{22}$$

From which we get $\|x\| = \|x^*\|$.