Completely bounded Maps and Operator Algebras
Chapter 14: An Operator Space Bestiary
Solutions

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14.1 Let V be a normed space, X an operator space and let $\varphi : X \to \mathrm{MIN}(V)$, $\psi : \mathrm{MAX}(V) \to X$ be linear maps. Prove that $\|\varphi\| = \|\varphi\|_{cb}$ and $\|\psi\| = \|\psi\|_{cb}$. Deduce that $\mathrm{MIN}(V)$ and $\mathrm{MAX}(V)$ are homogeneous.

Solution: Let $(x_{i,j}) \in M_n(X)$ with $\|(x_{i,j})\| \leq 1$. Recall that if g is a linear functional on an operator space, then g is completely bounded and $\|g\|_{cb} = \|g\|$. Consider,

$$\begin{aligned}
\|\varphi_n((x_{i,j}))\| &= \sup\{\|(f(\varphi(x_{i,j})))\| \,:\, f \in V_1^*\} && (1)\\
&= \sup\{\|(f \circ \varphi)_n((x_{i,j}))\| \,:\, f \in V_1^*\} && (2)\\
&\leq \sup \|(f \circ \varphi)_n\| && (3)\\
&= \|f \circ \varphi\| && (4)\\
&\leq \|f\| \, \|\varphi\| && (5)\\
&\leq \|\varphi\| && (6)
\end{aligned}$$

We may assume that $\|\psi\| \leq 1$. We begin by noting that

$$\|(v_{i,j})\|_{M_n(\mathrm{MAX}(V))} = \sup\{\|(\varphi(v_{i,j}))\| \,:\, \varphi : V \to B(\mathcal{H}), \|\varphi\| \leq 1\}. \qquad (7)$$

To prove this, note that the equation in (7) defines an operator space norm on V which is larger than $\|\|_{\mathrm{MAX}(V)}$. These must be equal, since $\|\|_{\mathrm{MAX}(V)}$ is the largest operator space norm on V. By Ruan's theorem there exists a complete isometry $\rho : X \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Note that the map $\rho \circ \psi : V \to B(\mathcal{H})$ is contractive. Therefore,

$$\begin{aligned}
\|(\psi(v_{i,j}))\|_X &= \|\rho_n(\psi(v_{i,j}))\| && (8)\\
&= \|((\rho \circ \psi)(v_{i,j}))\| && (9)\\
&\leq \|(v_{i,j})\|_{M_n(\mathrm{MAX}(V))}. && (10)
\end{aligned}$$

Hence $\|\psi_n\| \leq 1$ and $\|\psi\| = \|\psi\|_{cb}$

14.2 Let V be a normed space, X an operator space and let $\varphi : \mathrm{MIN}(V) \to X$. Prove that $\|\varphi\|_{cb} \leq \alpha(V)\,\|\varphi\|$.

Solution: We may assume $\alpha(V) < \infty$. Define $j : \mathrm{MIN}(V) \to \mathrm{MAX}(V)$ by $j(v) = v$, and $\psi : \mathrm{MAX}(V) \to X$ by $\psi(x) = \varphi(x)$. Note,

$$\|\psi(v)\| = \|\varphi(v)\| \leq \|\varphi\|\,\|v\|_{\mathrm{MIN}(V)} \leq \|\varphi\|\,\|v\|_{\mathrm{MAX}(V)}. \qquad (11)$$

Therefore, ψ is bounded and $\|\psi\| \leq \|\varphi\|$. By Exercise 14.1 ψ is completely bounded and $\|\psi\|_{cb} = \|\psi\|$. Since $\varphi = \psi \circ j$ we see that φ is completely bounded. Now,

$$\|\varphi\|_{cb} = \|\psi \circ j\|_{cb} \leq \|\psi\|_{cb}\,\|j\|_{cb} = \alpha(V)\,\|\psi\| \leq \alpha(V)\,\|\varphi\|. \qquad (12)$$

14.3 (Zhang) Let $\mathbb{F}_n$ denote the free group on n generators

$$\{u_1^{(n)}, \ldots, u_n^{(n)}\}. \tag{13}$$

Prove that the maps $\varphi : \mathrm{MAX}(\ell_n^1) \to C_u^*(\mathbb{F}_n)$ and $\psi : \mathrm{MAX}(\ell_n^1) \to C_u^*(\mathbb{F}_{n-1})$ given by

$$\varphi((\lambda_1, \ldots, \lambda_n)) = \lambda_1 u_1^{(n)} + \ldots + \lambda_n u_n^{(n)}, \tag{14}$$

and

$$\psi((\lambda_1, \ldots, \lambda_n)) = \lambda_1 u_1^{(n-1)} + \ldots + \lambda_{n-1} u_{n-1}^{(n-1)} + \lambda_n I, \tag{15}$$

are complete isometries.

Solution: Let $e_1, \ldots, e_n \in \ell_n^1$ denote the standard basis, let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \ell_n^1$ and let $(v_{i,j}) = \sum_{k=1}^n A_k \otimes e_k \in M_m(\mathrm{MAX}(\ell_n^1))$. We have $\varphi_m((v_{i,j})) = \sum_{k=1}^n A_k \otimes u_k^{(n)}$. Assume that $\rho : \ell_n^1 \to B(\mathcal{H})$. From,

$$\|\rho(\lambda)\| = \left\|\sum_{k=1}^n \lambda_k \rho(e_k)\right\| \leq \sum_{k=1}^n |\lambda_k| \, \|\rho(e_k)\|, \tag{16}$$

it follows that ρ is contractive if and only if $\rho(e_k)$ is a contraction on $\mathcal{H}$ for $k = 1, \ldots, n$. By definition, and the observation just made, we have,

$$\begin{aligned} \left\|\sum_{k=1}^n A_k \otimes e_k\right\|_{\mathrm{MAX}(\ell_n^1)} &= \sup\left\{\left\|\sum_{k=1}^n A_k \otimes \rho(e_k)\right\|_{B(\mathcal{H}^{(m)})} : \rho : \ell_n^1 \to B(\mathcal{H}), \|\rho\| \leq 1\right\} && (17) \\ &= \sup\left\{\left\|\sum_{k=1}^n A_k \otimes T_k\right\|_{B(\mathcal{H}^{(m)})} : T_1, \ldots, T_n \in B(\mathcal{H}), \|T_k\| \leq 1\right\} && (18) \end{aligned}$$

Now,

$$\left\|\sum_{k=1}^n A_k \otimes u_k^{(n)}\right\|_{C^*(\mathbb{F}_n)} = \sup\left\{\left\|\sum_{k=1}^n A_k \otimes U_k\right\| : U_1, \ldots, U_n \in U(\mathcal{H}), \mathcal{H} \text{ a Hilbert space}\right\} \tag{19}$$

where $U(\mathcal{H})$ denotes the unitary group of $B(\mathcal{H})$. If $\rho : \ell_n^1 \to B(\mathcal{H})$ then $\rho(e_k)$ is a contraction and so the supremum on the right side of (19) is smaller than the right side of (17).

By exercise 5.4, given any n contractions $T_1, \ldots, T_n$ on $\mathcal{H}$ we can dilate $T_1, \ldots, T_n$ to n unitaries $U_1, \ldots, U_n$ on $\mathcal{K}$. Therefore,

$$\left\|\sum_{k=1}^n A_k \otimes T_k\right\|_{B(\mathcal{H}^{(n)})} \leq \left\|\sum_{k=1}^n A_k \otimes U_k\right\|_{B(\mathcal{K}^{(n)})}. \tag{20}$$

Hence, (17) and (19) are equal.

If $(v_{i,j}) = \sum_{k=1}^n A_k \otimes e_k \in M_m(\mathrm{MAX}(\ell_n^1))$, then $\psi_m((v_{i,j})) = \sum_{k=1}^{n-1} A_k \otimes u_k^{(n-1)} + A_n \otimes I$. The norm of ψ_m is given by,

$$\left\|\sum_{k=1}^{n-1} A_k \otimes u_k^{(n-1)} + A_n \otimes I\right\|_{C^*(\mathbb{F}_{n-1})} \tag{21}$$

By the universal property of $C^*(\mathbb{F}_n)$ there is a $*$-homomorphism $\pi : C^*(\mathbb{F}_n) \to C^*(\mathbb{F}_{n-1})$ such that $u_k^{(n)} \mapsto u_{k-1}^{(n-1)}$ for $k = 1, \dots, n-1$ and $u_n^{(n)} \mapsto 1$. Similarly there is a $*$-homomorphism $\sigma : C^*(\mathbb{F}_{n-1})$ to $C^*(\mathbb{F}_n)$ such that$u_k^{(n-1)} \to u_n^{(n)*} u_k^{(n)}$. We have,

$$(\pi \circ \sigma)(u_k^{(n-1}) = \pi(u_n^{(n)*} u_k^{(n)}) = u_k^{(n-1)}. \tag{22}$$

Therefore σ is one-one and a $*$-isomorphism. Now,

$$\begin{aligned}
\left\| \sum_{k=1}^{n-1} A_k \otimes u_k^{(n-1)} + A_n \otimes I \right\|_{C^*(\mathbb{F}_{n-1})} &= \left\| \sum_{k=1}^{n-1} A_k \otimes u_n^{(n)*} u_k^{(n)} + A_n \otimes u_n^{(n)} u_n \right\|_{C^*(\mathbb{F}_n)} && (23)\\
&= \left\| \sum_{k=1}^{n} (I_n \otimes u_n^{(n)})(A_k \otimes u_k^{(n)}) \right\|_{C^*(\mathbb{F}_n)} && (24)\\
&= \left\| (I_n \otimes u_n^{(n)}) \sum_{k=1}^{n} A_k \otimes u_k^{(n)} \right\|_{C^*(\mathbb{F}_n)} && (25)\\
&= \left\| \sum_{k=1}^{n} A_k \otimes u_k^{(n)} \right\|_{C^*(\mathbb{F}_n)}. && (26)
\end{aligned}$$

(26) is due to the fact that $I_n \otimes u_n^{(n)}$ is a unitary in $M_m(C^*(\mathbb{F}_n))$.

14.4 Prove that the maps $\varphi : \mathrm{MIN}(\ell_n^1) \to C(\mathbb{T}^n)$ and $\psi : \mathrm{MIN}(\ell_n^1) \to C(\mathbb{T}^{n-1})$ given by

$$\varphi((\lambda_1, \dots, \lambda_n)) = \lambda_1 z_1 + \ldots + \lambda_n z_n \tag{27}$$

and

$$\psi((\lambda_1, \dots, \lambda_n)) = \lambda_1 z_1 + \ldots + \lambda_{n-1} z_{n-1} + \lambda_n \tag{28}$$

are complete isomteries.

Solution: Let $(v_{i,j}) \in M_m(\ell_n^1)$, $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \ell_m^2$ and $z = (z_1, \dots, z_n) \in \mathbb{T}^n$. If $\mu = (\mu_1, \dots, \mu_n) \in \ell_n^1$, then

$$\|\mu\|_{\ell_n^1} = \sup_{z_1, \dots, z_n \in \mathbb{T}} |\mu_1 z_1 + \ldots + \mu_n z_n| . \tag{29}$$

It is a consequence of the triangle inequality that the right hand side of (29) is smaller than the left hand side. To prove the reverse inequality we simply choose $z_j \in \mathbb{T}$, $j = 1, \dots, n$ such that $\mu_j z_j = |\lambda_j|$

for $j = 1, \ldots, n$. To see that φ is a complete isometry we compute,

$$
\begin{aligned}
\|(\varphi(v_{i,j}))\|_{M_m(C(\mathbb{T}^n))} &= \sup_{z\in\mathbb{T}^n} \|(\varphi(v_{i,j})z)\|_{M_m} && (30)\\
&= \sup_{z\in\mathbb{T}^n} \sup_{\|\alpha\|_2\le 1, \|\beta\|_2\le 1} |\langle(\varphi(v_{i,j})z)\alpha, \beta\rangle| && (31)\\
&= \sup_{z\in\mathbb{T}^n} \sup_{\|\alpha\|_2\le 1, \|\beta\|_2\le 1} \left|\sum_{i,j=1}^{m} \varphi(v_{i,j})(z)\overline{\alpha_i}\beta_j\right| && (32)\\
&= \sup_{z\in\mathbb{T}^n} \sup_{\|\alpha\|_2\le 1, \|\beta\|_2\le 1} \left|\sum_{i,j=1}^{m} \varphi(v_{i,j})(z)\alpha_i\beta_j\right| && (33)\\
&= \sup_{\|\alpha\|_2\le 1, \|\beta\|_2\le 1} \sup_{z_1,\ldots,z_n\in\mathbb{T}} \left|\sum_{k=1}^{n} \left(\sum_{i,j=1}^{m} \alpha_i\lambda_{i,j}^{(k)}\beta_j\right) z_k\right| && (34)\\
&= \sup_{\|\alpha\|_2\le 1, \|\beta\|_2\le 1} \left\|\sum_{i,j=1}^{m} \alpha_i v_{i,j}\beta_j\right\|_{\ell_n^1} && (35)\\
&= \|(v_{i,j})\|_{M_m(\mathrm{MIN}(\ell_n^1))}. && (36)
\end{aligned}
$$

To establish that ψ is a complete isometry we need only prove the following analogue of (29):

$$
\|\mu\|_{\ell_n^1} = \sup_{z_1,\ldots,z_{n-1}\in\mathbb{T}} |\mu_1 z_1 + \ldots + \mu_{n-1}z_{n-1} + \mu_n|. \tag{37}
$$

If we choose $w \in \mathbb{T}$ such that $w\mu_n = |\mu_n|$ and for $j = 1, \ldots, n-1$, pick $z_j \in \mathbb{T}$ such that $z_j\mu_j = w^{-1}|\mu_j|$, then

$$
\begin{aligned}
\|\mu\|_{\ell_n^1} &= |z_1 w\mu_1 + \ldots + z_{n-1}w\mu_{n-1}| && (38)\\
&= |w|\,|z_1\mu_1 + \ldots + z_{n-1}\mu_{n-1} + \mu_n| && (39)\\
&= |z_1\mu_1 + \ldots + z_{n-1}\mu_{n-1} + \mu_n|. && (40)
\end{aligned}
$$

This implies (37).

14.5 Prove Proposition 14.7.

Solution: We endow $M_{m,n}(CB(E,F))$ with the norm it inherits through its identification with $CB(E, M_{m,n}(F))$. Let $\Phi = (\varphi_{i,j}) \in M_{m,n}(CB(E,F))$ and let $X = (x_{k,l}) \in M_r(E)$. Let $A \in M_{p,m}$ and $B \in M_{n,q}$.We have,

$$
\begin{aligned}
(A\Phi B)_r((x_{k,l})) &= ((A\Phi B)(x_{k,l})) && (41)\\
&= (A\Phi(x_{k,l})B) && (42)\\
&= \underbrace{(A \oplus \ldots \oplus A)}_{r\text{ times}} \Phi_r((x_{k,l})) \underbrace{(B \oplus \ldots \oplus B)}_{r\text{ times}}. && (43)
\end{aligned}
$$

Hence,

$$
\|(A\Phi B)_r)((x_{k,l}))\| \le \|A\|\,\|\Phi_r((x_{k,l}))\|\,\|B\|. \tag{44}
$$

It follows,

$$\begin{aligned}
\|A\Phi B\|_{CB(E,F)} &= \|A\Phi B\|_{cb} && (45)\\
&= \sup_{r\geq 1}\|(A\Phi B)_r\| && (46)\\
&= \sup_{r\geq 1}\|A\|\,\|\Phi_r\|\,\|B\| && (47)\\
&= \|A\|\,\|\Phi\|_{cb}\,\|B\| && (48)\\
&= \|A\|\,\|\Phi\|_{CB(E,F)}\,\|B\|\,. && (49)
\end{aligned}$$

Let $\Phi \in M_{m,n}(CB(E,F))$ and $\Psi \in M_{p,q}(CB(E,F))$. Now,

$$\begin{aligned}
(\Phi\oplus\Psi)_r((x_{k,l})) &= ((\Phi\oplus\Psi)(x_{k,l})) && (50)\\
&= \left(\begin{bmatrix} \Phi(x_{k,l}) & 0 \\ 0 & \Psi(x_{k,l}) \end{bmatrix}\right). && (51)
\end{aligned}$$

A permutation of the rows of this last matrix shows that it has the same norm as

$$\begin{bmatrix} \Phi_r((x_{k,l})) & 0 \\ 0 & \Psi_r((x_{k,l})) \end{bmatrix}. \tag{52}$$

Hence,

$$\begin{aligned}
\|\Phi\oplus\Psi\|_{CB(E,F)} &= \|\Phi\oplus\Psi\|_{cb} && (53)\\
&= \sup_{r\geq 1}\|(\Phi\oplus\Psi)_r\| && (54)\\
&= \sup_{r\geq 1}\max\{\|\Phi_r\|\,,\|\Psi_r\|\} && (55)\\
&= \max\{\sup_{r\geq 1}\|\Phi_r\|\,,\sup_{r\geq 1}\|\Psi_r\|\} && (56)\\
&= \max\{\|\Phi\|_{CB(E,F)}\,,\|\Psi\|_{CB(E,F)}\}. && (57)
\end{aligned}$$

Thus, $CB(E,F)$ is an operator space.

14.8 Let $\gamma : C_n \to R_n$ be the map $\gamma(\sum_{i=1}^n \lambda_i E_{i,1}) = \sum_{i=1}^n \lambda_i E_{1,i}$. Prove that γ is an isometry and $\|\gamma\|_{cb} = \left\|\gamma^{-1}\right\|_{cb} = \sqrt{n}$.

Solution:As γ is the transpose map γ is an isometry. Let $C = (C_{i,j}) \in M_m(C_n)$ where

$$C_{i,j} = \left[\begin{array}{c|c} \lambda_1^{(i,j)} & \\ \vdots & 0 \\ \lambda_n^{(i,j)} & \end{array}\right]. \tag{58}$$

Let

$$R_{i,j} = \gamma(C_{i,j}) = \left[\begin{array}{ccc} \lambda_1^{(i,j)} & \dots & \lambda_n^{(i,j)} \\ \hline & 0 & \end{array}\right] \tag{59}$$

By a canonical shuffle we see that

$$\|(C_{i,j})\| = \left\|\left[\begin{array}{c|c} A_1 & \\ \dots & 0 \\ A_n & \end{array}\right]\right\|, \tag{60}$$

where $(A_k)_{i,j} = \lambda_k^{(i,j)}$ and $A_k \in M_m$. Similarly

$$\|(R_{i,j})\| = \left\| \left[\begin{array}{ccc} A_1 & \cdots & A_n \\ \hline & 0 & \end{array} \right] \right\|. \tag{61}$$

Now,

$$\begin{aligned}
\left\| \left[\begin{array}{c|c} A_1 & \\ \cdots & 0 \\ A_n & \end{array} \right] \right\|^2 &= \left\| \left[\begin{array}{c|c} A_1 & \\ \cdots & 0 \\ A_n & \end{array} \right]^* \left[\begin{array}{c|c} A_1 & \\ \cdots & 0 \\ A_n & \end{array} \right] \right\| && (62) \\
&= \left\| \left[\begin{array}{cc} \sum_{i=1}^n A_i^* A_i & 0 \\ 0 & 0 \end{array} \right] \right\| && (63) \\
&= \left\| \sum_{i=1}^n A_i^* A_i \right\|, && (64)
\end{aligned}$$

and,

$$\begin{aligned}
\left\| \left[\begin{array}{ccc} A_1 & \cdots & A_n \\ \hline & 0 & \end{array} \right] \right\|^2 &= \left\| \left[\begin{array}{ccc} A_1 & \cdots & A_n \\ \hline & 0 & \end{array} \right] \left[\begin{array}{c|c} A_1^* & \\ \cdots & 0 \\ A_n^* & \end{array} \right] \right\| && (65) \\
&= \left\| \left[\begin{array}{cc} \sum_{i=1}^n A_i A_i^* & 0 \\ 0 & 0 \end{array} \right] \right\| && (66) \\
&= \left\| \sum_{i=1}^n A_i A_i^* \right\|, && (67)
\end{aligned}$$

Since $\|A_k A_k^*\| \le \|\sum_{i=1}^n A_i A_i^*\|$ we see that,

$$\begin{aligned}
\left\| \sum_{i=1}^n A_i^* A_i \right\| &\le \sum_{i=1}^n \|A_i\|^2 && (68) \\
&= \sum_{i=1}^n \|A_i A_i^*\| && (69) \\
&\le n \left\| \sum_{i=1}^n A_i A_i^* \right\| && (70)
\end{aligned}$$

A similar argument shows that $\|\sum_{i=1}^n A_i A_i^*\| \le n \, \|\sum_{i=1}^n A_i^* A_i\|$. Therefore

$$\|\gamma_m\| \le \sqrt{n} \text{ and } \left\|\gamma_m^{-1}\right\| \le \sqrt{n} \tag{71}$$

for all $m \ge 1$.

To prove equality in (71) we consider $E \in M_n(C_n)$ given by

$$E = \left[\begin{array}{c|c} e_1 & \\ \vdots & 0 \\ e_n & \end{array} \right]. \tag{72}$$

by permuting rows we see that $\|E\| = \sqrt{n}$. Note that a permutation of the rows of $\gamma_n(E)$ brings it to the form

$$\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}. \tag{73}$$

Therefore $\|\gamma_n(E)\| = 1$. It follows that $\left\|\gamma_n^{-1}\right\| \geq \sqrt{n}$. By considering

$$E = \left[\begin{array}{ccc} e_1^T & \cdots & e_n^T \\ \hline & 0 & \end{array}\right], \tag{74}$$

we get $\|\gamma_n\| \geq \sqrt{n}$. Combining these with the inequalities in (71) we get

$$\|\gamma\|_{cb} = \left\|\gamma^{-1}\right\|_{cb} = \sqrt{n}. \tag{75}$$