Completely Bounded Maps and Operator Algebras
Chapter 15: Injective Envelopes
Solutions to Exercises

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15.1 Verify that an operator space I is injective in $\mathcal{O}_1$ if and only if every completely bounded map into I has a completely bounded extension of the same completely bounded norm.

Solution: Let $\varphi : E \to I$ be a completely bounded map and let $E \subseteq F$. We may assume that $\varphi \neq 0$. If every completely bounded map into I has an extension of the same completely bounded norm, then trivially every completely contractive map into I has a a completely contractive extension.

For the converse let $\tilde{\varphi} = \varphi / \|\varphi\|_{cb}$. Then $\|\tilde{\varphi}\|_{cb} = 1$ and since I is injective in $\mathcal{O}_1$, there exists $\tilde{\psi} : F \to I$ such that $\|\tilde{\psi}\|_{cb} = 1$. Let $\psi = \|\varphi\|_{cb} \tilde{\psi}$. It is straightforward that ψ extends φ and $\|\psi\|_{cb} = \|\varphi\|_{cb} \|\tilde{\psi}\|_{cb} = \|\varphi\|_{cb}$.

15.2 Let $E \subseteq B(\mathcal{H})$ be an operator space. Prove that E is injective if and only if there exists a completely contractive map $\varphi : B(\mathcal{H}) \to E$ such that $\varphi(e) = e$ for an $e \in E$.

Solution: Assume first that E is injective. Extend the identity map on E to a map $\varphi : B(\mathcal{H}) \to E$. The map φ is completely contractive, because the identity is a complete isometry. If $e \in E$, then

$$\varphi(e) = \mathrm{id}_E(e) = e. \tag{1}$$

For the converse let $F \subseteq G$ be operator spaces and let $\psi : F \to E$ be a complete contraction. By composing with the inclusion map $j : E \to B(\mathcal{H})$ we get a complete contraction $j \circ \psi : F \to B(\mathcal{H})$. By Wittstock's theorem there exists a complete contraction $\rho : G \to B(\mathcal{H})$ such that $\rho|_F = j \circ \psi$. The map $\varphi \circ \rho : G \to E$ is completely contractive and

$$(\varphi \circ \rho)(f) = \varphi(j(\psi(f))) = \psi(f), \tag{2}$$

since ρ extends $j \circ \psi$ and φ fixes E.

15.3 Let $M \subseteq B(\mathcal{H})$ be an operator space and let $\varphi : B(\mathcal{H}) \to B(\mathcal{H})$ be an M-projection with p_φ a minimal M-seminorm. Assume that $u \in B(\mathcal{H})$ is a unitary that commutes with M. Prove that if $\gamma(x) = u^*\varphi(x)u$ then p_γ is a minimal M-seminorm and γ is a projection onto $u^*\varphi(B(\mathcal{H}))u$. Prove that if $\psi(x) = \varphi(u^*xu)$ then p_ψ is a minimal M-seminorm and ψ is a (possibly different) projection onto $\varphi(B(\mathcal{H}))$.

Solution: If $x \in M$, then

$$\gamma(x) = u^*\varphi(x)u = u^*xu = u^*ux = x. \tag{3}$$

This shows that γ fixes M. Assume that p is a seminorm such that $p \leq p_\gamma$. Note that

$$p_\gamma(x) = \|u^*\varphi(x)u\| = \|\varphi(x)\| = p_\varphi(x), \tag{4}$$

since u is unitary. Now,

$$p(x) \leq p_\gamma(x) = p_\varphi(x), \tag{5}$$

and so by the minimailty of φ we have that $p = p_\varphi = p_\gamma$. This shows that p_γ is a minimal M-seminorm and from theorem 15.4 that γ is an M-projection. The range of γ is clearly a subset of $u^*\varphi(B(\mathcal{H}))u$. The fact that,

$$\gamma(B(\mathcal{H})) = u^*\varphi(B(\mathcal{H}))u, \tag{6}$$

follows directly from $\gamma(x) = u^*\varphi(x)u$.

Next consider the map ψ. If $x \in M$, then,

$$\psi(x) = \varphi(u^*xu) = \varphi(u^*ux) = \varphi(x) = x, \tag{7}$$

and so ψ is an M-map. Assume that $p \leq p_\psi$ and define a $\tilde{p}(x) = p(uxu^*)$. Now,

$$\tilde{p}(x) = p(uxu^*) \leq \|\psi(uxu^*)\| = \|\varphi(x)\| = p_\varphi(x). \tag{8}$$

From the minimality of p_φ we have that $\tilde{p} = p_\varphi$. Hence,

$$p(x) = \tilde{p}(u^*xu) = \|\varphi(u^*xu)\| = \|\psi(x)\| = p_\psi(x). \tag{9}$$

Once again by Theorem 15.4 ψ is a minimal M-projection

15.5 Let $\varphi : B(\mathcal{H}) \to B(\mathcal{H})$ be a unital completely positive map that fixes the compacts. Prove that φ is necessarily the identity map.

Solution: As every operator is a linear combination of at most 4 positive operators, it is enough to prove that φ fixes every positive operator. In fact it is enough to show that φ fixes every invertible positive operator. To see this note that $P+\varepsilon I$ is invertible for $\varepsilon > 0$ and $P \geq 0$. If $\varphi(P+\varepsilon I) = P+\varepsilon I$, then $\varphi(P) = P$, since φ is unital.

We will need a Cholesky-type decomposition for positive operators on a separable Hilbert space which we now describe.

Cholesky Decomposition Let $P = (p_{i,j})_{i,j=1}^\infty \in B(\ell^2)$ be positive and invertible.

$$P = \begin{bmatrix} p_{1,1} & A \\ A^* & B \end{bmatrix}, \tag{10}$$

with $p_{1,1} > 0$. By the Cholesky decomposition described in exercise 3.9 we see that the operator

$$P_1 = \begin{bmatrix} 0 & 0 \\ 0 & B - p_{1,1}^{-1}A^*A \end{bmatrix} = P - \begin{bmatrix} p_{1,1}^{-1/2} & 0 \\ A^* & 0 \end{bmatrix} \begin{bmatrix} p_{1,1}^{-1/2} & A \\ 0 & 0 \end{bmatrix} \geq 0 \tag{11}$$

Let $R_1 = \begin{bmatrix} p_{1,1}^{-1/2} & A \\ 0 & 0 \end{bmatrix}$. If we now repeat this process we get a sequence R_k of rank one operators such that

$$0 \leq P - \sum_{k=1}^n R_k^* R_k \leq \begin{bmatrix} 0 & 0 \\ 0 & B_n \end{bmatrix}, \tag{12}$$

where B_n denotes the compression of P to the subspace

$$\mathcal{M}_n := \{(x_k)_{k=1}^\infty \,:\, x_1 = \ldots = x_n = 0\}. \tag{13}$$

Therefore,

$$\lim_{n\to\infty}\sum_{k=1}^{n} R_k^* R_k = P, \tag{14}$$

in the SOT.

Let P be positive. Then there exists a sequence of operators $\{R_k\}_{k=1}^{\infty}$ such that R_k is rank one, $K_n = \sum_{k=1}^{n} R_k^* R_k \leq P$ for all $n \geq 1$ and $\sum_{k=1}^{\infty} R_k^* R_k = P$ in the strong operator topology.

Since φ fixes the compacts we see that $K_n = \varphi(K_n) \leq \varphi(P)$. Hence,

$$\langle Px, x\rangle = \lim_{n\to\infty} \langle K_n x, x\rangle \leq \langle \varphi(P)x, x\rangle\,, \tag{15}$$

which says that $P \leq \varphi(P)$. By choosing a constant α such that $\alpha I - P$ is positive and using the arguments above we see that

$$\varphi(\alpha I - P) \geq \alpha I - P. \tag{16}$$

As φ is unital we get $\varphi(P) \leq P$.

We now deal with the non-separable case. Let x be in $\mathcal{H}$. Let $\mathcal{M}$ be the closed linear span of $\{P^n x : n \geq 0\}$, which is a reducing subspace for P. With respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$ the matrix of P has the form

$$P = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \tag{17}$$

with $A, B \geq 0$. By the Cholesky decomposition we see that there is a sequence $\{K_n\}$ of positive compact operators on $\mathcal{M}$ which are bounded above by A and converge to A in the SOT. We have,

$$\begin{bmatrix} K_n & 0 \\ 0 & 0 \end{bmatrix} = \varphi \begin{bmatrix} K_n & 0 \\ 0 & 0 \end{bmatrix} \leq \varphi \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \varphi(P). \tag{18}$$

Therefore,

$$\langle Px, x\rangle = \langle Ax, x\rangle = \lim_{n\to\infty} \langle K_n x, x\rangle \leq \langle \varphi(P)x, x\rangle\,. \tag{19}$$

Arguing as in the separable case we see that $\varphi(P) = P$.

15.6 Let $\mathcal{S} \subseteq B(\mathcal{H})$ be an operator system which contains the compacts. Prove that $I(\mathcal{S}) = B(\mathcal{H})$.

Solution: Recall that $I(\mathcal{S})$ is the image of a completely positive map $\varphi : B(\mathcal{H}) \to B(\mathcal{H})$ which fixes $\mathcal{S}$. This map therefore fixes the compacts and is unital. By the previous exercise, φ must be the identity map. Therefore, $I(\mathcal{S}) = \varphi(B(\mathcal{H})) = B(\mathcal{H})$.

15.7 Let $\mathcal{A} \subseteq B(\ell^2)$ be the algebra of upper triangular operators. Prove that $C_e^*(\mathcal{A}) = B(\ell^2)$.

Solution: Let P be a positive invertible operator. Note that by the Cholesky decomposition described in exercise 15.5, there exists a sequence of operators $K_n = \sum_{k=1}^{n} R_k^* R_k$ such that $K_n \to P$ in the SOT. Therefore $P = U^*U$ where U is an upper-triangular operator whose rows are the R_k's. It follows that U is bounded and so the C^*-algebra generated by the upper-triangular operators contains every positive invertible operator. Therefore, $C^*(\mathcal{A}) = B(\ell^2)$. By Hamana's theorem (Theorem 15.16) there exists a onto $*$-homomorphism $\pi : B(\ell^2) \to C_e^*(\mathcal{A})$ which fixes $\mathcal{A}$. If π is not one-one, the kernel of π, which is a two-sided ideal in $B(\ell^2)$ must contain $K(\mathcal{H})$, the ideal of compact operators, and so $\pi(K(\mathcal{H})) = \{0\}$. Since $\mathcal{A}$ contains non-zero compact operators we see that π is a $*$-isomorphism. We have

$$\pi(P) = \pi(U^*U) = \pi(U)^*\pi(U) = U^*U = P. \tag{20}$$

Thus, π fixes all the positive invertible operators and must therefore be the identity map. Hence, $C_e^*(\mathcal{A}) = B(\ell^2)$.

15.8 Prove that $\partial_S A(\mathbb{D}) = \mathbb{T}$.

Solution: As $A(\mathbb{D}) \subseteq C(\mathbb{T})$ we see that $\partial_S A(\mathbb{D}) \subseteq \mathbb{T}$. We know that the restriction map $r : C(\mathbb{T}) \to C(\partial_S A(\mathbb{D}))$ given by $r(f) = f|_{\partial_S A(\mathbb{D})}$ is isometric on $A(\mathbb{D})$. For each $z \in \mathbb{T}$, let $f_z(e^{i\theta}) = z + e^{i\theta}$. It is clear that $\|f_z\|_{C(\mathbb{T})} = 2$ and that this is attained at $e^{i\theta} = z$. Also, $\left|f_z(e^{i\theta})\right| < 2$ at all other points on the circle. Therefore, r cannot be an isometry on $A(\mathbb{D})$ unless $\partial_S A(\mathbb{D}) = \mathbb{T}$.

15.9 Prove that if $\mathcal{A} \subseteq M_n$, then $C_e^*(\mathcal{A})$ is finite-dimensional.

Solution: By Hamana's theorem there exists an onto, $*$-homomorphism $\pi : C^*(\mathcal{A}) \to C_e^*(\mathcal{A})$. Being a subspace of M_n, $C^*(\mathcal{A})$ is finite-dimensional and $C^*(\mathcal{A})/\ker(\pi)$ is $*$-isomorphic to $C_e^*(\mathcal{A})$. Hence, the C^*-envelope of $\mathcal{A}$ is finite-dimensional.

15.12 Let $\{u_1, \ldots, u_n\}$ denote the unitaries in $C^*(\mathbb{F}_n)$ and let M denote the $(n+1)$-dimensional subspace spanned by these unitaries and the identity I. If $\mathcal{A} \subseteq M_2(C^*(\mathbb{F}_n))$ denotes the $(n+3)$-dimensional operator algebra

$$\mathcal{A} = \left\{ \begin{bmatrix} \lambda I & x \\ 0 & \mu I \end{bmatrix} : \lambda, \mu \in \mathbb{C}, x \in M \right\}, \tag{21}$$

then prove that $C_e^*(\mathcal{A}) = M_2(C^*(\mathbb{F}_n))$. Thus, a finite-dimensional operator algerba can have an infinite-dimensional C^*-envelope.

Solution: By Hamana's theorem there exists an onto $*$-homomorphism from $C^*(\mathcal{A}) \to C_e^*(\mathcal{A})$ such that $\pi(a) = a$. We will show that the C^*-algebra generated by $\mathcal{A}$ is all of $M_2(C^*(\mathbb{F}_n))$ and that this homomorphism is one-one.

We see that

$$\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}^* = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \in C^*(\mathcal{A}). \tag{22}$$

If u is one of the generating unitaries, then

$$\begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & u \end{bmatrix} \in C^*(\mathcal{A}). \tag{23}$$

Therefore, $M_2(C^*(\mathbb{F}_n)) = C^*(\mathcal{A})$.

By matrix factorings similar to the ones used above we can check that π fixes all of $M_2(C^*(\mathbb{F}_n))$. Therefore, π is a $*$-isomorphism and $M_2(C^*(\mathbb{F}_n)) = C^*(\mathcal{A})$.