Completely Bounded Maps and Operator Algebras
Chapter 16: Abstract Operator Algebras
Solutions to Exercises

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16.1 Let C_n denote n-dimensional column Hilbert space. Prove that $I(S_{C_n}) \cong M_{n+1}$ in such a way that

$$I_{11}(C_n) = M_l(C_n) \cong M_n \text{ and } I_{22} = M_r(C_n) \cong \mathbb{C}.$$

Solution: Since C_n inherits its operator space structure from $M_{n,1}$, the operator system

$$S_{c_n} := \left\{ \begin{bmatrix} \lambda I_n & x \\ y^* & \mu \end{bmatrix} : \lambda, \mu \in \mathbb{C}, x, y \in C_n \right\},$$

is naturally identified as an operator system in M_{n+1}. M_{n+1} is an injective C^*-algebra containing S_{C_n} and we claim that it is minimal. Suppose that E is injective and $S_{C_n} \subseteq E \subseteq M_{n+1}$. Then there is a completely positive map $\varphi : M_{n+1} \to E \subset M_{n+1}$ which fixes S_{c_n}. We claim that the only completely positive map $\varphi : M_{n+1} \to M_{n+1}$ which fixes S_{C_n} is the identity map on M_{n+1}. To prove this we will show that φ is the Schur multiplier induced by the matrix of all 1's.

If φ fixes S_{C_n}, then φ is unital and fixes the matrix units $E_{j,n+1}$ and $E_{n+1,j}$ for all $j = 1, \ldots, n+1$. Let $1 \leq j \leq n+1$ and note that

$$\begin{aligned} \varphi(E_{j,j}) &= \varphi(E_{j,n+1}E_{n+1,j}) && (1) \\ &= \varphi(E^*_{n+1,j}E_{n+1,j}) && (2) \\ &\geq \varphi(E_{n+1})^*\varphi(E_{n+1,j}) && (3) \\ &= E^*_{n+1,j}E_{n+1,j} = E_{j,j}. && (4) \end{aligned}$$

Now,

$$\varphi(I - E_{j,j}) = \varphi(\sum_{k \neq j} E_{k,k}) \geq \sum_{k \neq j} E_{k,k} = I - E_{j,j}. \tag{5}$$

It follows that $\varphi(E_{j,j}) \leq E_{j,j}$. We have established that φ fixes the diagonal matrices and is consequently a Schur multiplier. Let J be the $(n+1) \times (n+1)$ matrix all of whose entries are 1. The matrix which induces the multiplier is given by $\varphi(J)$. Let $R = \sum_{j=1}^{n+1} E_{n+1,j}$. We have,

$$\varphi(J) = \varphi(R^*R) \geq \varphi(R)^*\varphi(R) = R^*R = J. \tag{6}$$

From the fact that φ in unital it follows that the diagonal entries of $\varphi(J)$ are all 1. The matrix $\varphi(J) - J$ is positive and has zeros on the diagonal and must therefore be 0. Hence $\varphi(J) = J$ and so φ is the identity map. Having established that $I(S_{C_n}) \cong M_{n+1}$ we see that $I_{11}(C_n)$ is the $n \times n$ top left corner of M_{n+1} which is M_n. Similarly $I_{22}(C_n)$ is the bottom right 1×1 corner which we identify with $\mathbb{C}$.

16.2 Prove, by showing that the Blecher-Ruan-Sinclair axioms are met, that the matrix-normed algebra $(\mathcal{P}_n, \|\cdot\|_{u,k})$, introduced in Chapter 5, is a unital operator algebra.

Solution: Note that the universal operator algebra has matrix norms defined only for square matrices. It is easy to check that $\|\cdot\|_k$ satisfies the conditions of exercise 13.2 and so $\mathcal{P}_n$ is a matrix normed space.

We now claim that it is an L^∞ matrix normed space. If $\{T_1, \ldots, T_n\}$ are n commuting contractions on a Hilbert space $\mathcal{H}$ then note that the operator matrices $(p_{i,j}(T_1, \ldots, T_n))_{i,j=1}^m$ are elements of the concrete operator algebra $B(\mathcal{H}^{(m)})$.

If $A, B \in M_k$ and $X = (p_{i,j}) \in M_k(\mathcal{P}_n)$, then

$$\begin{aligned}
\|AXB\| &= \sup \left\| \left(\sum_{l,m=1}^{k} a_{i,l} p_{l,m}(T_1, \ldots, T_n) b_{m,j} \right) \right\| && (7)\\
&\leq \sup \|(a_{i,j}I)\| \, \|(p_{i,j}((T_1, \ldots, T_n))\| \, \|(b_{i,j}I)\| && (8)\\
&= \|A\| \, \|(p_{i,j})\|_{u,k} \, \|B\| && (9)
\end{aligned}$$

which shows that $(\mathcal{P}_n, \|\cdot\|_{u,k})$ is a matrix normed space. Let $(p_{i,j}) \in M_k(\mathcal{P}_n)$ and $(q_{s,t}) \in M_m(\mathcal{P}_n)$.

$$\begin{aligned}
\|(p_{i,j}) \oplus (q_{s,t})(T_1, \ldots, T_n)\| &= \|(p_{i,j}(T_1, \ldots, T_n)) \oplus (q_{s,t}(T_1, \ldots, T_n))\| && (10)\\
&= \max\{\|(p_{i,j}(T_1, \ldots, T_n))\|, \|(q_{s,t}(T_1, \ldots, T_n))\|\}. && (11)
\end{aligned}$$

where the last equality follows from the fact that we are in a concrete operator algebra, as $(p_{i,j}(T_1, \ldots, T_n))$ and $(q_{s,t}(T_1, \ldots, T_n))$ are operators on a Hilbert space. It follows that,

$$\begin{aligned}
\|(p_{i,j}) \oplus (q_{s,t})\|_{u,k+m} &= \sup \|(p_{i,j}) \oplus (q_{s,t})(T_1, \ldots, T_n)\| && (12)\\
&= \sup\{\max\{(p_{i,j}(T_1, \ldots, T_n)), (q_{s,t}(T_1, \ldots, T_n))\}\} && (13)\\
&= \max\{\sup\{(p_{i,j}(T_1, \ldots, T_n))\}, \sup\{(q_{s,t}(T_1, \ldots, T_n))\}\} && (14)\\
&= \max\{\|(p_{i,j})\|_{u,k}, \|(q_{s,t})\|_{u,m}\}. && (15)
\end{aligned}$$

It remains to show that the multiplication on $M_k(\mathcal{P}_n)$ is contractive. We have,

$$\begin{aligned}
\|(p_{i,j})(q_{i,j})\|_{u,k} &= \sup \|(p_{i,j}(T_1, \ldots, T_n))(q_{i,j}(T_1, \ldots, T_n))\| && (16)\\
&\leq \sup \|(p_{i,j}(T_1, \ldots, T_n))\| \, \|(q_{i,j}(T_1, \ldots, T_n))\| && (17)\\
&\leq \|(p_{i,j})\|_{u,k} \, \|(q_{i,j})\|_{u,k} && (18)
\end{aligned}$$

Finally the polynomial that is identically 1 is the unit for $(\mathcal{P}_n, \|\cdot\|_{u,k})$.

16.3 Let $\mathcal{A}$ be a unital operator algebra and let J be a non-trivial 2-sided ideal in $\mathcal{A}$. Prove that the algebra $\mathcal{A}/J$ equipped with the quotient operator space structure is an operator algebra.

Solution: We have already seen that $\mathcal{A}/J$ is an operator space and the unit is $\mathbf{1} + J$. It is enough to show that the multiplication is contractive. Let $(a_{i,j}), (b_{i,j}) \in M_m(\mathcal{A}/J)$. Choose $(x_{i,j}), (y_{i,j}) \in J$ such that $\|(a_{i,j}) + (x_{i,j})\|_m \leq \|(a_{i,j} + J)\|_m + \varepsilon$ and $\|(b_{i,j}) + (y_{i,j})\|_m \leq \|(b_{i,j} + J)\|_m + \varepsilon$. It follows that,

$$\begin{aligned}
(\|(a_{i,j} + J)\|_m + \varepsilon)(\|(b_{i,j} + J)\|_m + \varepsilon) &\geq \|(a_{i,j}) + (x_{i,j})\|_m \, \|(b_{i,j}) + (y_{i,j})\|_m && (19)\\
&\geq \|((a_{i,j}) + (x_{i,j}))((b_{i,j}) + (y_{i,j}))\|_m && (20)\\
&= \left\| (a_{i,j})(b_{i,j}) + \left(\sum_{k=1}^{m} x_{i,k} b_{k,j} + a_{i,k} y_{k,j} + x_{i,k} y_{k,j} \right) \right\|_m && (21)\\
&\geq \|(a_{i,j})(b_{i,j}) + J\|_m && (22)
\end{aligned}$$

By letting $\varepsilon \to 0$ we get our result.