Fall 2011. SM 223. Practice Test 1. Solutions

1. Show that $2x^2 + 2y^2 - 4x - 6y = 0$ is a circle. Give its center and radius.

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   Complete the square to get $x^2-2x+1+y^2-3y+9/4 = 13/4$. This gives $(x-1)^2+(y-3/2)^2 = 13/4$. The center is $(1, 3/2)$ and the radius is $\sqrt{13}/2$.

2. Compute the vector projection of the vector $\langle 2, 3, 4\rangle$ onto to the vector $\langle -1, 3, 1\rangle$ and compute the angle between these two vectors.

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   Let $\mathbf{b} = \langle 2, 3, 4\rangle$ and let $\mathbf{a} = \langle -1, 3, 1\rangle$. We have,

$$\text{proj}_{\mathbf{a}}(\mathbf{b}) = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{11}{\sqrt{11}^2}\langle -1, 3, 1\rangle = \langle -1, 3, 1\rangle.$$

   The angle is given by

$$\theta = \cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}\right) = \cos^{-1}\left(\frac{11}{\sqrt{11}\sqrt{29}}\right) = 52^\circ.$$

3. Find all unit vectors perpendicular to both $2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$ and $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

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   To get a perpendicular vector we compute $\mathbf{a}\times\mathbf{b} = \langle 2, 3, -6\rangle \times \langle 1, -1, 2\rangle = \langle 0, -10, -5\rangle$. We now rescale this to be unit length by dividing through by the length, which is $\sqrt{125} = 5\sqrt{5}$. Hence, the two perpendicular unit vectors are

$$\pm\langle 0, \frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\rangle.$$

4. Find the parametric equation for the line of intersection of the planes $x + y - z = 2$ and $x + 3y + z = -1$.

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   To get a point on the intersection we need to solve these equations. We have $x + y - z - 2 = 0 = x + 3y + z + 1$. This gives $2y + 2z = -3$. Setting $z = 0$, we get $y = -3/2$. Substituting this back into the equation for the first plane we get $x = -7/2$. A point on the line is $(-7/2, -3/2, 0)$.

   The direction can be obtained by computing the cross product of the normal vectors. We have $\langle 1, 1, -1\rangle \times \langle 1, 3, 1\rangle = \langle 4, -2, 2\rangle$.

   The equation of the line is $\langle 7/2 + 4t, -3/2 - 2t, 2t\rangle$.

5. Find the equation of the plane that is perpendicular to the line $x = 2 + t$, $y = 3 - 4t$ and $z = 6 + 3t$ and contains the point $(1, 1, 1)$.

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   The equation of the plane is $\mathbf{n}\cdot\langle x, y, z\rangle = \mathbf{n}\cdot\langle 1, 1, 1\rangle$, where $\mathbf{n}$ is the normal. Since the plane is perpendicular to the line, we can set $\mathbf{n} = \langle 1, -4, 3\rangle$.

   Hence, the equation of the plane is $x - 4y + 3z = 0$.

6. Sketch the surface $z = x^2 + y^2$. Indicate clearly the intersections with the coordinate planes and the axes.

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   When we set $x = 0$ we get $z = y^2$ which is a parabola. The same is true when we set $x = 0$, we get a parabola. To figure out the cross-section we take a slice at height z to get $x^2 + y^2 = z$. This is a circle of radius $\sqrt{z}$.

   We did the sketch in class.

7. Explain why the planes $x + y + z = 3$ and $-x - y - z = 19$ do not intersect.

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   These two planes do not intersect since they are parallel. They are parallel because their normal vectors are parallel. We have $\langle 1, 1, 1\rangle = -\langle -1, -1, -1\rangle$.