Fall 2011. SM 223. Practice Test 2 (Solutions)

1. Find the parametric equation of the line segment that connects the points $(0, 3)$ and $(2, 1)$.

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   The direction is $\langle 2, 1\rangle - \langle 0, 3\rangle = \langle 2, -2\rangle$. The equation of the line segment is $\langle 0, 3\rangle + t\langle 2, -2\rangle$. So $x = 2t$, $y = 3 - 2t$ for $0 \le t \le 1$.

2. Let $\mathbf{r}(t) = \langle \cos(t), \sin(t), \cos^2(t)\rangle$. find the equation of the tangent line to this curve at the point $(\sqrt{3}/2, 1/2, 3/4)$.

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   First we find the value of t for which $\mathbf{r}(t) = (\sqrt{3}/2, 1/2, 3/4)$. We have $\sqrt{3}2 = \cos(t)$ and so $t = \pi/6$ or $t = 11\pi/6$. Now $\sin(t) = \frac{1}{2}$ and so $t = \pi/6$. Next the derivative of the function $\mathbf{r}$ gives the direction of the tangent line. The derivative is $\mathbf{r}'(t) = \langle -\sin(t), \cos(t), -2\sin(t)\cos(t)\rangle$. Substituting the value of $t = \pi/6$ we get the direction of the line as $\mathbf{v} = \langle -\sin(\pi/6), \cos(\pi/6), -2\sin(\pi/6)\cos(\pi/6)\rangle = \langle -1/2, \sqrt{3}/2, -\sqrt{3}/2\rangle$. Hence, the equation of the tangent line is

$$x = \frac{\sqrt{3}}{2} - \frac{t}{2}, y = \frac{1}{2} + \frac{\sqrt{3}}{2}t, z = \frac{3}{4} - \frac{\sqrt{3}}{2}t.$$

3. Find the length of the curve $\mathbf{r}(t) = \langle t, e^t, e^{-t}\rangle$ for $0 \le t \le 1$.

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   We have $\mathbf{r}'(t) = \langle 1, e^t, -e^{-t}\rangle$ and $\|\mathbf{r}'(t)\| = \sqrt{1 + e^{2t} + e^{-2t}}$. The arc length can be computed by

$$L = \int_0^1 \sqrt{1 + e^{2t} + e^{-2t}}\, dt = 2.122$$

4. Captain Kirk is stranded on the prison planet of Rura Penthe. Without his phaser he must resort to throwing a rock to knock out the power grid. His target is 20m away and he can throw the rock at 60 km/h. His trusted friend Spock does a quick calculation and says "Jim, you need to throw this rock at precisely ...". What angle does Spock indicate? [Hint: no knowledge of Star Trek is required to do this problem]

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   The speed in m/s is $60 \times 1000/3600 = 50/3$. The formula for the range of a projectile is $R = \frac{v^2}{g}\sin(2\theta)$. We have $R = 20$, $g = 9.8$ and we solve to get $\theta = 22.43$.

5. Let $f(x, y) = \sqrt{x - 4y^2}$. Find the domain and range of the function f. Determine the level curves and sketch a contour map.

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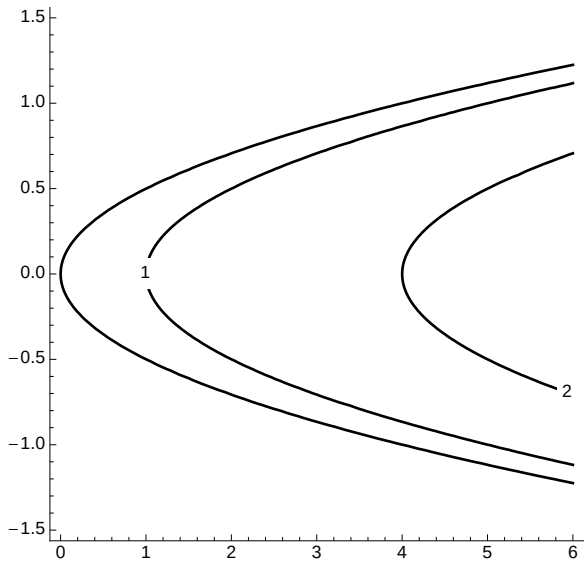
   The domain is the set of points (x, y) for which $x - 4y^2 \ge 0$. The curve $x = 4y^2$ is a parabola and the domain is the set of points that are on the "inside" of the parabola.

   The range of the function is $\{z : z \ge 0\}$.

   To find the level curves we set $k = \sqrt{x - 4y^2}$ which simlifies to $x = 4y^2 + k^2$. This is a family of parabolas. We sketch the level curves for $k = 0, 1, 2$.



6. Does the function $u(x, y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ satisfy the equation $u_{xx} + u_{yy} = 0$? Calculations are required to justify your answer.

We have $u_x = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $u_{xx} = -\sin(x)\cos(y) - \cos(x)\sin(y)$, $u_y = -\sin(x)\sin(y) + \cos(x)\cos(y)$ and $u_{yy} = -\sin(x)\cos(y) - \cos(x)\sin(y)$. We see that $u_{xx} = u_{yy}$ and so $u_{xx} - u_{yy} = 0$. The answer is no.

7. Suppose that terrain near a hill is given by the equation $h(x, y) = (x^2 + y)e^{-x^2-y^2}$. You are at the point $(0, 1/2)$. In which direction should you walk if you want

   (a) the toughest climb;

   (b) the easiest descent;

   (c) to stay at the same elevation.

   In each case compute the rate of change of the elevation.

Recall that the gradient points in the direction of greatest rate of change. We have $\nabla f = \langle 2x(1 - x^2 - y)e^{-x^2-y^2}, (1 - 2x^2y - 2y^2)e^{-x^2-y^2}\rangle$. Substituting $x = 0$, $y = 1/2$ we get $\nabla f(0, 1/2) = \langle 0, \frac{1}{2}e^{-1/4}\rangle$. If we divide by the length then we get a unit vector in the direction of the gradient, which is $\mathbf{u} = \langle 0, 1\rangle$. The direction of greatest increase (toughest climb) is $\langle 0, 1\rangle$. The direction fo greatest decrease is $-\nabla f = \langle 0, -1\rangle$. To stay at the same elevation we travel perpendicular to the gradient, i.e., travel in the direction of $\langle 1, 0\rangle$.

The maximum rate of change is $\|\nabla f\| = \frac{1}{2}e^{-1/4} \approx 0.3894$. The minimum rate of change is $-\|\nabla f\| \approx -0.3894$ and the rate of change in the perpendicular direction is 0.

8. Suppose that a particle is traveling along the curve $\mathbf{r}(t) = \langle te^t, te^{-t}, t\rangle$. Find the velocity, speed, and acceleration of the particle when $t = 2$.

We have $\mathbf{v}(t) = \mathbf{r}'(t) = \langle (1+t)e^t, (1-t)e^{-t}, 1\rangle$, $\mathbf{a}(t) = \mathbf{v}'(t) = \langle (2+t)e^t, (t-2)e^{-t}, 0\rangle$. At $t = 2$, the velocity is $\langle 3e^2, -e^{-2}, 1\rangle$ and $\mathbf{a}(t) = \langle 4e^2, 0, 0\rangle$. The speed is $\sqrt{(3e^2)^2 + (-e^{-2})^2 + 1} \approx 22.19$.

9. Let $P = (1, 2)$, $Q = (2, -1)$ and $R = (1, 1)$. Suppose the directional derivative of the function f at P in the direction to Q is -1 and that the directional derivative of f at P in the direction to R is 1. Find the gradient of f at the point P.

In unit vector in the direction from P to Q is $\mathbf{u} = \langle 1, -3\rangle/\sqrt{10}$. Using the formula $D_{\mathbf{u}}(f) = (\nabla f) \cdot \mathbf{u}$ we get $-1 = \frac{1}{\sqrt{10}} f_x - \frac{3}{\sqrt{10}} f_y$. Now from P to R the unit vector is $\mathbf{u} = \langle 0, -1\rangle$. Again using the formula for the directional derivative we get $1 = -f_y$. Hence, $f_y = -1$ and $f_x = -\sqrt{10} - 3$. The gradient is $\nabla f = \langle -\sqrt{10} - 3\rangle -1$.