Fall 2011. SM 223. Practice Test 3

This is a practice test. No claim is made that the real test will have the same or similar problems on it. The only claim is that the practice test covers the same concepts and topics as the real test. **Have you done and reviewed the homework? You probably should do that first.**

1. Let $f(x, y, z) = e^{x^2-y^2} - z^2$.

   (a) Find the tangent plane to the level surface $f(x, y, z) = 0$ at the point $(2, -2, 1)$. Write your answer in the form $z = ax + by + c$ for some constants a, b, c.

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   $\nabla f = \langle 2xe^{x^2-y^2}, -2ye^{x^2-y^2}, -2z\rangle$. At the point $(2, -2, 1)$ we get $\mathbf{n} = nablaf(2, -2, 1) = \langle 4, 4, -2\rangle$. The equation of the tangent plane is $\mathbf{n} \cdot \langle x, y, z\rangle = \mathbf{n} \cdot \langle 2, -2, 1\rangle$ which gives $4x + 4y - 2z = -2$ or $z = 2x + 2y + 1$.

   (b) Use this or any other method to find the linear approximation of the function $z = g(x, y) = e^{(x^2-y^2)/2}$ at the point $(2, -2)$.

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   From the previous part we have $z^2 = e^{x^2-y^2}$ which gives $z = e^{(x^2-y^2)/2}$. The linear approximation to this function at the point $(2, -2)$ is $z = 2x + 2y + 1$.

   (c) Use this linear approximation to estimate the value of $g(2.01, -1.98)$.

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   We have $z = 2(2.01) + 2(-1.98) + 1 = 1.06$.

2. Suppose that $z = f(x, y)$ and that $x = s^2 - t^2$, $y = s^2 + t^2$ for some function f. Use the table below to compute $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ when $s = 1$, $t = -2$.

(x, y)	$f(x, y)$	$f_x(x, y)$	$f_y(x, y)$	$f_{xx}(x, y)$	$f_{yy}(x, y)$	$f_{xy}(x, y)$
$(1, -2)$	1	2	3	4	5	6
$(-3, 5)$	7	8	9	10	11	12
$(2, -4)$	13	14	15	16	17	18

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By the chain rule we have

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

When $s = 1$ and $t = -2$ we get $x = -3$, $y = 5$. $\frac{\partial x}{\partial s} = 2s = 2$ and $\frac{\partial y}{\partial s} = 2s = 2$. Hence

$$\frac{\partial z}{\partial s} = 2f_x(-3, 5) + 2f_y(-3, 5) = 2(8) + 2(9) = 34.$$

For $\frac{\partial z}{\partial t}$ we need $\frac{\partial x}{\partial t} = -2t = 4$ and $\frac{\partial y}{\partial t} = 2t = -4$. This gives

$$\frac{\partial z}{\partial t} = 4(8) - 4(9) = -4.$$

3. Find the local maxima and minima for the function $f(x, y) = xy + \dfrac{1}{x} + \dfrac{1}{y}$. For each critical point determine whether it is a local max, local min or saddle point and give the value of the function at that point. Present your answer clearly.

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We have $f_x = y - \frac{1}{x^2}$, $f_y = x - \frac{1}{y^2}$, $f_{xx} = \frac{2}{x^3}$, $f_{yy} = \frac{2}{y^3}$ and $f_{xy} = 1$.

Set $f_x = 0$ to get $y = \frac{1}{x^2}$. Substitute into $f_y = 0$ to get $x - x^4 = 0$. This gives $x = 0, 1$. Note that $x = 0$ is can be eliminated since it is not in the domain of f. Hence, the only ciritical point is $(1, 1)$.

At this point $D = f_{xx}(1,1)f_{yy}(1,1) - f_{xy}(1,1)^2 = 4 - 1 = 3 > 0$ and $f_{xx} = 2 > 0$. Hence, $(1, 1)$ is a minimum. The value of the function at this point is 3.

4. Find the maximum and minimum values of the functions $f(x, y, z) = x^3y^3z^3$ subject to the constraint $x^2 + y^2 + z^2 = 3$. Give all points at which the maximum and minimum value is attained and compute the maximum and minimum values.

Lagrange equations

$$f_x = 3x^2y^3z^3 = \lambda g_x = 2\lambda x \tag{1}$$
$$f_y = 3x^3y^2z^3 = \lambda g_y = 2\lambda y \tag{2}$$
$$f_z = 3x^3y^3z^2 = \lambda g_z = 2\lambda z \tag{3}$$

Multiply the equations by x, y, z respectively to get $\lambda x^2 = \lambda y^2 = \lambda z^2$. Note that $\lambda = 0$ implies that at least one of the variables x, y, z is 0, in which case the function is 0.

Assume $\lambda \neq 0$ and we get $x^2 = y^2 = z^2$. Using the constraint gives us the equation $3x^2 = 3$, hence $x = \pm 1$, $y = \pm 1$ and $z = \pm 1$. This gives rise to eight ciritical points of the form $(\pm 1, \pm 1, \pm 1)$. At four of these points the value of the function is 1 and at the other four it is -1. Hence, the maximum is 1, and the minimum is -1.

5. Find the maximum volume of an open-top box whose surface area is 1 m^2.

Let x, y, z denote the sides of the box. The area of the box is $xy + 2zx + 2yz = 1$, (remember it has an open-top). The volume is $f(x, y, z) = xyz$.

Using Lagrange multipliers we get

$$f_x = yz = \lambda g_x = \lambda(y + 2z) \tag{4}$$
$$f_y = xz = \lambda g_y = \lambda(x + 2z) \tag{5}$$
$$f_z = xy = \lambda g_z = 2\lambda(x + y) \tag{6}$$

We can assume that $\lambda \neq 0$ since $x, y, z > 0$. Multiplying the three equations above by x, y, z respectively and setting them equal gives $xy + 2zx = xy + 2zy = 2xz + 2zy$. Hence, $2zx = 2zy$ and we get $x = y$. Also $xy = 2xz$ and $y = 2z$. Using the constraint we get $x^2 + x^2 + x^2 = 1$ and so $x = \frac{1}{\sqrt{3}}$. We have the sides of the box are $\frac{1}{\sqrt{3}}$, $\frac{1}{\sqrt{3}}$, and $\frac{1}{2\sqrt{3}}$. The volume of the box is $\frac{1}{6\sqrt{3}}$.