SM311O First Exam
17 Feb 1999

1. (**40 points**) Determine

   (a) f_x and f_y if $f(x,y) = x^2y^3 - x^3y^2 + \ln(x^2+y^2)$.

   (b) f_{xx} if $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$.

   (c) $\nabla \cdot \mathbf{v}$ if $\mathbf{v} = \langle \sin x \cos y, -\cos x \sin y \rangle$.

   (d) $\nabla \phi$ if $\phi(x,y) = \frac{y^2}{x}$.

   (e) $\nabla \times \mathbf{v}$ if $\mathbf{v} = \langle y+x, -x+y^2, 0 \rangle$.

   (f) $\nabla \times \nabla f$ if f is an arbitrary function of. x, y and z.

   (g) $\nabla(\text{div } \mathbf{v})$ if $\mathbf{v} = \langle \sin 2x, \cos 3y \rangle$

   (h) $\nabla \cdot (\nabla g)$ if $g = x^2 + y^2 + z^2$.

2. (a) (**10 points**) Let $\psi = 2x^2 - y^2 + 3xy$ be the stream function of a flow. Determine the velocity of the fluid particle located at $(2, -3)$.

   (b) (**10 points**) Consider the velocity field $\mathbf{v} = \langle x^2 - y^2, -2xy + x^3 \rangle$. Does $\mathbf{v}$ have a stream function? If yes, determine it.

3. (**20 points**) Let $\mathbf{v} = \langle y^3, 0 \rangle$.

   (a) Plot the velocity vectors for the points located at $(0,0)$, $(0,1)$ and $(0,2)$. Give an example of a current in the oceans that this flow may be modeling.

   (b) Compute the vorticity of this flow. Does every particle experience the same rate of spinning? Are there any particles that do not experience any sensation of spinning?

4. (**20 points**)

   (a) Prove the identity div $(\psi \mathbf{v}) = \psi$div $\mathbf{v} + \mathbf{v} \cdot \nabla \psi$, where ψ and $\mathbf{v}$ are arbitrary scalar and vector functions of x, y and z.

   (b) Use the above identity to determine div $(\psi \mathbf{v})$ if $\mathbf{v}$ is a velocity field of a 2-D incompressible steady–state flow and ψ is its stream function.

5. (**Bonus – 10 points**) Let ψ be the stream function of 2-D incompressible steady–state flow $\mathbf{v}$. Use your knowledge of the relationship between ψ and $\mathbf{v}$ to show that

$$\mathbf{v} = \nabla \times \psi \mathbf{k}$$

where $\mathbf{k} = \langle 0, 0, 1 \rangle$.