SM311O First Exam (Solutions)
17 Feb 1999

1. (**40 points**) Determine

   (a) f_x and f_y if $f(x,y) = x^2y^3 - x^3y^2 + \ln(x^2+y^2)$.
   Solution: $f_x = -3\,x^2\,y^2 + 2\,x\,y^3 + \frac{2\,x}{x^2+y^2}$, $f_y = -2\,x^3\,y + 3\,x^2\,y^2 + \frac{2\,y}{x^2+y^2}$.

   (b) f_{xx} if $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$.
   Solution: $f_{xx} = \frac{3\,x^2}{(x^2+y^2)^{\frac{5}{2}}} - (x^2+y^2)^{-\frac{3}{2}}$

   (c) $\nabla \cdot \mathbf{v}$ if $\mathbf{v} = \langle \sin x \cos y, -\cos x \sin y \rangle$.
   Solution: $\nabla \cdot f = \cos x \cos y - \cos x \cos y = 0$.

   (d) $\nabla \phi$ if $\phi(x,y) = \frac{y^2}{x}$.
   Solution: $\nabla \left(\frac{y^2}{x} \right) = \langle -\frac{y^2}{x^2}, \frac{2y}{x} \rangle$.

   (e) $\nabla \times \mathbf{v}$ if $\mathbf{v} = \langle y + x, -x + y^2, 0 \rangle$.
   Solution: $\nabla \times \langle y + x, -x + y^2, 0 \rangle = \langle 0, 0, -2 \rangle = -2\mathbf{k}$.

   (f) $\nabla \times \nabla f$ if f is an arbitrary function of. x, y and z.
   Solution: $\nabla \times \nabla f = \nabla \times \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} = \langle 0, 0, 0 \rangle$, because for smooth functions the order of differentiation is immaterial.

   (g) $\nabla(\text{div } \mathbf{v})$ if $\mathbf{v} = \langle \sin 2x, \cos 3y \rangle$
   Solution: $\nabla(\text{div } \langle \sin 2x, \cos 3y \rangle) = \nabla(2\cos 2x - 3\sin 3y) = \langle -4\sin 2x, -9\cos 3y \rangle$

   (h) $\nabla \cdot (\nabla g)$ if $g = x^2 + y^2 + z^2$.
   Solution: $\nabla \cdot (\nabla(x^2 + y^2 + z^2)) = \nabla \cdot \langle 2x, 2y, 2z \rangle = 6$.

2. (a) (**10 points**) Let $\psi = 2x^2 - y^2 + 3xy$ be the stream function of a flow. Determine the velocity of the fluid particle located at $(2, -3)$.
   Solution: Because $\mathbf{v} = \langle \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \rangle$, we have $\mathbf{v} = \langle -2y + 3x, -4x - 3y \rangle$ which, when evalauted at $(2, -3)$, yields $\langle -12, 1 \rangle$.

   (b) (**10 points**) Consider the velocity field $\mathbf{v} = \langle x^2 - y^2, -2xy + x^3 \rangle$. Does $\mathbf{v}$ have a stream function? If yes, determine it.
   Solution: First we determine the divergence of $\mathbf{v}$: div $\mathbf{v} = 2x - 2x = 0$, so $\mathbf{v}$ does have a stream function. To determine ψ we set up the equations

$$\frac{\partial \psi}{\partial y} = x^2 - y^2, \qquad -\frac{\partial \psi}{\partial x} = -2xy + x^3. \tag{1}$$

We begin by integrating the first equation with respect to y;

$$\psi(x, y) = x^2 y - \frac{y^3}{3} + f(x). \tag{2}$$

Next, we differentiate the above equation with respect to x

$$\frac{\partial \psi}{\partial x} = 2xy + f'(x)$$

and compare equate ψ_x from the latter term with ψ_x in (1). We conclude that $f'(x) = -x^3$, or $f(x) = -\frac{x^4}{4}$. Returning to (2), we find that

$$\psi(x, y) = x^2 y - \frac{y^3}{3} - \frac{x^4}{4}.$$

3. (**20 points**) Let $\mathbf{v} = \langle y^3, 0 \rangle$.

   (a) Plot the velocity vectors for the points located at $(0, 0)$, $(0, 1)$ and $(0, 2)$. Give an example of a current in the oceans that this flow may be modeling.
   Solution: Note that $\mathbf{v}(0, 0) = \langle 0, 0 \rangle$, $\mathbf{v}(0, 1) = \langle 1, 0 \rangle$, $\mathbf{v}(0, 2) = \langle 8, 0 \rangle$. See Figure 1. This type of $\mathbf{v}$ models shear flows, such as the one one may observe in body of water that is being sheared by wind.

   (b) Compute the vorticity of this flow. Does every particle experience the same rate of spinning? Are there any particles that do not experience any sensation of spinning?
   Solution: $\omega = \nabla \times \mathbf{v} = \langle 0, 0, -y^2 \rangle$. Two particles whose positions' y-coordinates are different experience different vorticities and tendencies of spinning. Particles located on the x-axis experience have 0 vorticity.

4. (**20 points**)

   (a) Prove the identity div $(\psi \mathbf{v}) = \psi$div $\mathbf{v} + \mathbf{v} \cdot \nabla \psi$, where ψ and $\mathbf{v}$ are arbitrary scalar and vector functions of x, y and z.
   Solution: L.H.S. = div $(\psi \mathbf{v})$ = div $(\langle \psi v_1, \psi v_2, \psi v_3 \rangle) = \frac{\partial(\psi v_1)}{\partial x} + \frac{\partial(\psi v_2)}{\partial y} + \frac{\partial(\psi v_3)}{\partial z} = \frac{\partial \psi}{\partial x} v_1 + \psi \frac{\partial v_1}{\partial x} + \frac{\partial \psi}{\partial y} v_2 + \psi \frac{\partial v_2}{\partial y} + \frac{\partial \psi}{\partial z} v_3 + \psi \frac{\partial v_3}{\partial z} = \psi(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}) + (v_1 \frac{\partial \psi}{\partial x} + v_2 \frac{\partial \psi}{\partial y} + v_3 \frac{\partial \psi}{\partial z}) = \psi$div $\mathbf{v} + \mathbf{v} \cdot \nabla \psi$ = R.H.S.

   (b) Use the above identity to determine div $(\psi \mathbf{v})$ if $\mathbf{v}$ is a velocity field of a 2-D incompressible steady–state flow and ψ is its stream function.
   Solution: Since $\mathbf{v}$ is incompressible, then div $\mathbf{v} = 0$. Also, $\mathbf{v}$ has a stream function so that $v_1 = \frac{\partial \psi}{\partial y}$ and $v_2 = -\frac{\partial \psi}{\partial x}$. Hence,

$$\text{div } \psi \mathbf{v} = \psi \text{div } \mathbf{v} + \nabla \psi \cdot \mathbf{v} = 0 + \langle \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y} \rangle \cdot \langle \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \rangle = \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \psi}{\partial x} = 0.$$

5. (**Bonus – 10 points**) Let ψ be the stream function of 2-D incompressible steady–state flow $\mathbf{v}$. Use your knowledge of the relationship between ψ and $\mathbf{v}$ to show that

$$\mathbf{v} = \nabla \times \psi \mathbf{k}$$

where $\mathbf{k} = \langle 0, 0, 1 \rangle$.

Solution: Note that

$$\mathbf{v} = \langle \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \rangle. \tag{3}$$

On the other hand, $\nabla \mathbf{k} = \nabla \times \langle 0, 0, \psi \rangle = \langle \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \rangle$, which is equal to (3).

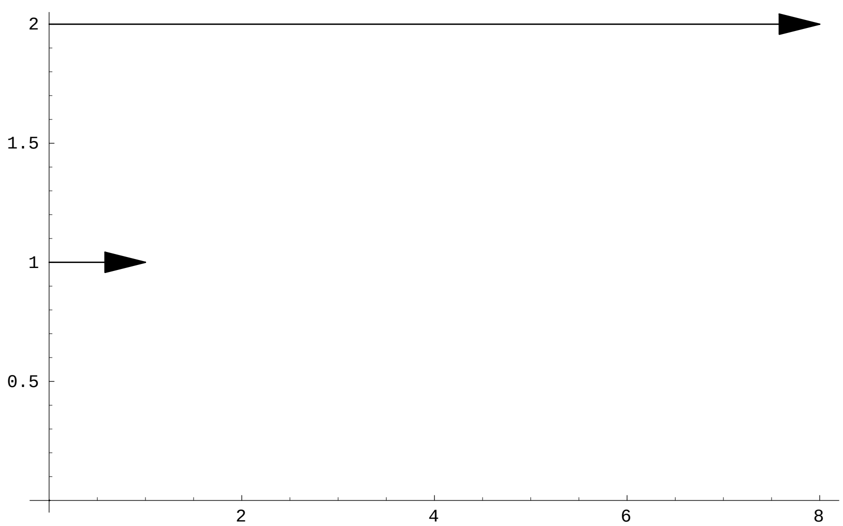

Figure 1: Sample vectors for the velocity field $\mathbf{v} = \langle y^3, 0 \rangle$.