**SM311O Second Exam (Solutions)**
**30 Oct 1998**

1. Let $\psi = x^2 + 3y^2 - 2xy$ be the stream function of a flow.

   (a) Determine the velocity field associated with $\psi$.
   **Solution**: Let $\psi = \langle x^2 + 3y^2 - 2xy \rangle$. Since $v_1 = \frac{\partial \psi}{\partial y}$ and $v_2 = -\frac{\partial \psi}{\partial x}$, we have
   $$v_1 = 6y - 2x, \quad v_2 = -2x + 2y.$$

   (b) Determine the line integral of the velocity field along the straight line that connects the points $(1, 2)$ to $(2, -1)$.
   **Solution**: The line connecting $(1, 2)$ and $(2, -1)$ has equation $y = -3x+5$. Let $\mathbf{r}(t) = \langle t, -3t + 5 \rangle$. Then
   $$\int_C \mathbf{v} \cdot d\mathbf{r} = \int_1^2 \langle 6(-3t + 5) - 2t, -2t + 2(-3t + 5) \rangle \cdot \langle 1, -3 \rangle \, dt = 6.$$

   (c) Determine the circulation of this flow around a circle of radius one centered at the origin.
   **Solution**: Let $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$. Then the line integral reduces to
   $$\int_0^{2\pi} \langle -2\cos t + 6 \sin t, -2\cos t + 2 \sin t \rangle \cdot \langle -\sin t, \cos t \, dt,$$
   which equals $-8\pi$.

2. Let $\mathbf{v} = \langle y \cos x, 2y + \sin x \rangle$.

   (a) Determine whether this velocity field has a potential $\phi$. If yes, find $\phi$.
   **Solution**: The curl of $\mathbf{v}$ must be zero if this vector field is going to have a potential. It is easy to check that $\nabla \times \mathbf{v} = \mathbf{0}$. Let $\phi$ be defined by $\mathbf{v} = \nabla \phi$. Then $\phi_x = y \cos x$ and $\phi_y = 2y + \sin x$. Integrating $\phi_x$ with respect to $x$ yields
   $$\phi(x, y) = y \sin x + f(y),$$
   where $f$ is the "constant" of integration. Computing $\phi_y$ from the latter equation yields $\phi_y = \sin x + f'(y)$ which, ahen compared with $\phi_y = 2y + \sin x$, yields $f'(y) = 2y$. So $f(y) = y^2$ and thus
   $$\phi(x, y) = y \sin x + y^2.$$

(b) Determine the circulation of this flow around a circle of radius 1 centered at the origin.

**Solution**: The answer is zero because $\mathbf{v}$ has a well-defined potential and the line integration is around a closed curve.

3. Let $\mathbf{v} = \langle y, 0 \rangle$.

(a) Plot some sample velocity vectors for points located on the $y$-axis.

**Solution**: Typical velocity vectors are parallel to the $x$-axis with magnitudes increasing with increasing $|y|$.

(b) Determine the line integral of $\mathbf{v}$ along the ellipse $x^2 + 3y^2 = 5$ when $x > 0$ and $y > 0$.

**Solution**: Casting the equation of the ellipse in the form

$$\frac{x^2}{5} + \frac{3y^2}{5} = 1$$

suggests parametrizing this curve as

$$\mathbf{r}(t) = \sqrt{5}\cos t, \sqrt{\frac{5}{3}}\sin t\rangle, \quad t \in (0, \frac{\pi}{2}.$$

Then

$$\int_C \mathbf{v} \cdot d\mathbf{r} = -\frac{5}{\sqrt{3}}\, dt = \int_0^{\frac{\pi}{2}} \sin^2 t = -\frac{5\pi}{4\sqrt{3}}.$$

4. Let $f(x) = x$. Find the Fourier sine series of $f$ in the interval $(0, 3)$.

**Solution**: $x = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{3}$ where

$$a_n = \frac{(x, \sin\frac{n\pi x}{3})}{(\sin\frac{n\pi x}{3}, \sin\frac{n\pi x}{3})} = \frac{2}{3}\int_0^3 x \sin\frac{n\pi x}{3}\, dx = -6\frac{\cos n\pi}{n\pi}.$$

5. Consider the heat conduction initial–boundary value problem

$$u_t = 4u_{xx}, \quad u(x, 0) = x, \quad u(0, t) = u(3, t) = 0.$$

(a) Determine the solution to this problem (you may wish to use the result of your computations in Problem 4).

**Solution**: After applying separation of variables to $u_t = 4u_{xx}$ we get

$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-4n^2\pi^2 t/9} \sin\frac{n\pi x}{3}.$$

The initial condition $u(x,0) = x$ dictates that $a_n$ must be the Fourier Sine series of $f$ in the interval $(0,3)$, which was already computed in the previous problem. Therefore,

$$u(x,t) = -6\sum_{n=1}^{\infty} \frac{\cos n\pi}{n\pi} e^{-4n^2\pi^2 t/9} \sin \frac{n\pi x}{3}.$$

(b) Using only one term of the series solution in part a), determine how long it takes for the temperature of the middle of the bar to reach half of its initial value.

**Solution**: The first term of this series solution is

$$\frac{6}{\pi} e^{-4\pi^2 t/9} \sin \frac{\pi x}{3}.$$

After evaluating this expression at $x = \frac{3}{2}$ and setting the result equal to $\frac{3}{4}$ (according to the initial conditions, the temperature at $x = \frac{3}{2}$ is $\frac{3}{2}$), we find that $t = 0.213089$.

6. In solving for the normal modes of the laplacian in the rectangular region $D = \{(x,y)|\, 0 < x < 2, 0 < y < 3\}$, we have arrived at the general solution

$$\psi(x,y) = (c_1 \sin \lambda x + c_2 \cos \lambda x)(c_3 \sin \sqrt{\lambda - \mu^2} y + c_4 \cos \sqrt{\lambda - \mu^2} y).$$

Determine the constants $c_1$, $c_2$, $c_3$, $c_4$, $\lambda$, and $\mu$ so that $\psi$ is a nontrivial normal mode of the laplacian in $D$.

**Solution**: The boundary conditions at $x = 0$ and $y = 0$ are satisfied if $c_2$ and $c_4$ are zero (why?). Hence

$$\psi(x,y) = c \sin \lambda x \sin \sqrt{\lambda - \mu^2} y$$

where $c = c_1 c_3$. Now, $\psi(2,y) = 0$, which implies that $\sin 2\lambda = 0$ or $\lambda_n = \frac{n\pi}{2}$. Similarly, $\psi(x,3) = 0$, which imples that $\sin 3\sqrt{\lambda - \mu^2} = 0$. Hence,

$$3\sqrt{\lambda - \mu^2} = m\pi, \quad \text{or} \quad \mu^2 = \frac{m^2\pi^2}{9} - \frac{n\pi}{2}.$$

Thus

$$\psi_{mn}(x,y) = c_{mn} \sin \frac{n\pi x}{2} \sin \frac{m\pi y}{3}.$$