SM311O Second Exam (Solutions)
26 Mar 1999

1. (**30 points**) Let $\psi = 2x^2 + 3y^2 - xy$ be the stream function of a flow.

   (a) Determine the velocity field associated with ψ.
   Solution: Since $\mathbf{v} = \langle \frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x} \rangle$, we have
   $$\mathbf{v} = \langle 6y - x, -4x + y \rangle.$$

   (b) Determine the line integral of the velocity field along the straight line that connects the points $(-1, 2)$ to $(2, -1)$.
   Solution: We first find that the line's equation is $y = -x + 1$. Hence, it can be parametrized as
   $$\mathbf{r}(t) = \langle t, -t + 1 \rangle, \qquad t \in (-1, 2).$$
   Since $\int_C \mathbf{v} \cdot d\mathbf{r} = \int_{-1}^{2} \mathbf{v} \cdot \frac{d\mathbf{r}}{dt} dt$, we have
   $$\int_C \mathbf{v} \cdot d\mathbf{r} = \int_{-1}^{1} \langle 6(-t+1) - t, -4t + (-t+1) \rangle \cdot \langle 1, -1 \rangle dt = \int_{-1}^{2} 5 - 2t \, dt = 12.$$

   (c) Determine the circulation of this flow around a circle of radius 3 centered at the origin.
   Solution: A circle of radius 3 centered at the origin can be parametrized as
   $$\mathbf{r}(t) = \langle 3\cos t, 3\sin t \rangle, \qquad t \in (0, 2\pi).$$
   Hence,
   $$\oint_C \mathbf{v} \cdot d\mathbf{r} = \int_0^{2\pi} \langle 6(3\sin t) - 3\cos t, -4(3\cos t) + \sin t \rangle \cdot \langle -3\sin t, 3\cos t \rangle dt =$$
   $$\int_0^{2\pi} 9(-5 + \cos 2t + \sin 2t) \, dt = -90\pi.$$

2. (**15 points**) Let $\mathbf{v} = \langle 6x^2 + z, -z^2, x - 2yz \rangle$.

   (a) Determine whether this velocity field has a potential ϕ. If yes, find ϕ.
   Solution: First we determine the curl of $\mathbf{v}$ and see that $\nabla \times \mathbf{v} = \langle 0, 0, 0 \rangle$. Hence, $\mathbf{v}$ does have a potential ϕ. To determine ϕ (recall that $\nabla\phi = \mathbf{v}$) we construct the following set of equations
   $$\frac{\partial\phi}{\partial x} = 6x^2 + z, \quad \frac{\partial\phi}{\partial y} = -z^2, \quad \frac{\partial\phi}{\partial z} = x - 2yz. \tag{1}$$

We begin by integrating the first equation in (1) with respect to x to get

$$\phi = 2x^3 + xz + f(y,z). \tag{2}$$

Next, differentiate this ϕ with respect to y to get $\frac{\partial \phi}{\partial y} = \frac{\partial f}{\partial y}$. Comparing this result to the second equation in (1), we have

$$\frac{\partial f}{\partial y} = -z^2$$

which, when integrated with respect to y, yields

$$f(y,z) = -z^2 y + g(z).$$

Substituting this information in (2) yields

$$\phi = 2x^3 + xz - z^2 y + g(z). \tag{3}$$

Continuing with determining ϕ, differentiate (3) with respect z to get $\frac{\partial \phi}{\partial z} = x - 2zy + g'(z)$. Comparing this information with the third equation in (1) we have that $g'(z) = 0$, which results in $g(z) = \text{const}$. As is our custom, we set const. equal to zero. Thus

$$\phi = 2x^3 + xz - z^2 y. \tag{4}$$

(b) Determine the line integral of this flow along the parabola $y = x^2$ in the $z = 1$ plane from A to B where $A = (0,0,1)$ and $B = (2,4,1)$.

Solution: Because vector field $\mathbf{v}$ has a potential, its line integral $\int_C \mathbf{v} \cdot d\mathbf{r}$ equals the potential difference at the endpoints of C. So

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \phi|_B - \phi|_A = 14.$$

3. (**15 points**) Let $f(x) = x(1-x)$. Find the Fourier sine series of f in the interval $(0,3)$. Use the first nonzero term of the Fourier series and evaluate it at $x = \frac{1}{2}$. How much does this value differ from $f(\frac{1}{2})$?

Solution: Here $\phi_n(x) = \sin \frac{n\pi x}{3}$. Then

$$a_n = \frac{(x(1-x), \phi_n)}{(\phi_n, \phi_n)}.$$

But, $(\phi_n, \phi_n) = \int_0^3 \sin^2 \frac{n\pi x}{3}\, dx = \frac{3}{2}$. Also

$$(x(1-x), \phi_n) = \int_0^3 x(1-x) \sin \frac{n\pi x}{3}\, dx = \frac{54(1 - \cos n\pi) + 18n^2\pi^2 \cos n\pi}{n^3\pi^3} \quad \text{using Mathematica.}$$

Hence

$$a_n = \frac{36(1 - \cos n\pi) + 12n^2\pi^2 \cos n\pi)}{n^3\pi^3}.$$

In particular

$$a_1 = \frac{72 - 12\pi^2}{\pi^3}.$$

Let $F(x) = a_1 \sin \frac{\pi x}{3}$ where a_1 is given above. Then $F(\frac{1}{2}) = \frac{36-6\pi^2}{\pi^3} = -0.748804$ while $f(\frac{1}{2}) = 0.25$.

4. (**25 points**) Consider the wave equation initial–boundary value problem

$$u_{tt} = 4u_{xx}, \quad u(x,0) = x(1-x), \quad u_t(x,0) = 0, \quad u(0,t) = u(3,t) = 0.$$

(a) Explain in words what $u(t,x)$ and each term in the above equations represent.

Solution: The function $u(x,t)$ represents the displacement of an elastic string from equilibrium at poisition x and at time t. The expressions $u(x,0)$ and $u_t(x,0)$ are the initial profile and the initial velocity of the string, respectively. The quantities $u(0,t)$ and $u(3,t)$ correspond to the displacement of the string at its two ends. The length of the string is 3 and its material constant is 4.

(b) Determine the solution to this problem (you may wish to use the result of your computations in the previous problem).

Solution: Starting with separation of variables (which we have done in class many times and I will not repeat here), the general solution to $u_{tt} = 4u_{xx}$ is

$$u(x,t) = (c_1 \sin 2\lambda t + c_2 \cos 2\lambda t)(c_3 \sin \lambda x + c_4 \cos \lambda x).$$

We start by applying the boundary conditions: $u(0,t) = 0$ forces c_4 to be zero (why?) while $u(3,t) = 0$ is satisfied if we choose $\lambda = \frac{n\pi}{3}$, $n = 1, 2, 3, \ldots$. Hence, u takes the form

$$u(x,t) = (A \sin \frac{2n\pi t}{3} + B \cos \frac{2n\pi t}{3}) \sin \frac{n\pi x}{3}.$$

Allowing A and B to be dependent on n, the general solution to $u_{tt} = 4u_{xx}$, $u(0,t) = u(3,t) = 0$ is

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \sin \frac{2n\pi t}{3} + B_n \cos \frac{2n\pi t}{3}) \sin \frac{n\pi x}{3}.$$

The initial datum $u_t(x,0) = 0$ is satisfied only if $A_n = 0$ for all n. Since $u(x,0) = x(1-x)$, the coefficients B_n must be the Fourier coefficients of $x(1-x)$ in terms of the Fourier functions $\sin\frac{n\pi x}{3}$. Hence, $B_1 = \frac{72-12\pi^2}{\pi^3}$ from the previous problem. Thus

$$u(x,t) = \frac{72-12\pi^2}{\pi^3}\cos\frac{2\pi t}{3}\sin\frac{\pi x}{3}$$

is a first approximation to the solution.

(c) Using only one term of the series solution in part (a), determine how long it takes for the string to go through one oscillation.

Solution: The period of $\cos\frac{2\pi t}{3}$ is 3.

5. (**15 points**) Consider an incompressible fluid occupying the slab

$$D = \{(x,y,z)|0 \le z \le H\}.$$

Let $\mathbf{v} = \langle v_1(x,y,z), v_2(x,y,z), v_3(x,y,z)\rangle$ be the velocity field of a motion generated in D. Suppose that

$$v_1 = x^2y^2z - xy, \quad v_2 = 3x^2 + y^2,$$

everywhere in D and that the vertical component of the velocity, v_3, is measured to be

$$x + y$$

at the **bottom** of D, i.e., when $z = 0$. Determine v_3 everywhere in D. (Hint: What does incompressibility mean **mathematically**?)

Solution: Since the fluid is incompressible, its velocity field satisfies

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0.$$

Substitute the expressions for v_1 and v_2 into this equation to get

$$\frac{\partial v_3}{\partial z} = -2xy^2z + y.$$

Integrate this equation with respect to z from $z = 0$ to $z = z$ to get

$$v_3(x,y,z,t) = -xy^2z^2 + zy + f(x,y,t).$$

(What is f?) But $v_3(x,y,0,t) = x + y$. Hence, $f(x,y,t) = x + y$ (why?) So

$$v_3(x,y,z,t) = -xy^2z^2 + zy + x + y.$$