SM311O (Third Exam) – Solutions
23 APRIL 2001

1. (**20 Points**) Give parametrizations for the following curves or surfaces:

   (a) The straight line connecting $(1, 2, 0)$ to $(-1, 4, 0)$. **Solution**: $\mathbf{r}(t) = \langle t, 3 - t, 0\rangle$.

   (b) The ellipse $2x^2 + 3y^2 = 6$ and located in the plane $z = 1$. **Solution**: $\mathbf{r}(u, v) = \langle\sqrt{3}\cos t, \sqrt{2}\sin t, 1\rangle$ with $t \in (0, 2\pi)$.

   (c) The plane passing through the points $(1, 0, 0)$, $(0, -1, 0)$ and $(0, 0, 2)$. **Solution**: Plane's equation is $2x - 2y + z = 2$ so its parametrization is $\mathbf{r}(u, v) = \langle u, v, 2 - 2u + 2v\rangle$.

   (d) The surface $z = 1 - x^2 - y^2$ with the circular domain $x^2 + y^2 \leq 4$. **Solution**: $\mathbf{r}(u, v) = \langle u\cos v, u\sin v, 1 - u^2\rangle$, $u \in (0, 2)$ and $v \in (2\pi)$.

2. (**20 Points**) Compute the flux of the velocity field $\mathbf{v} = x\mathbf{k}$ through the surface S, the first quadrant of the disk of radius 3 in the xy-plane centered at the origin and with $x > 0$ and $y > 0$.

   Solution: Flux of velocity is defined by $\int\int_S \mathbf{v} \cdot d\mathbf{A}$. The surface S can be parametrized as $\mathbf{r}(u, v) = \langle u\cos v, u\sin v, 0\rangle$, $u \in (0, 3)$, $v \in (0, \frac{\pi}{2})$. $\mathbf{r}_u \times \mathbf{r}_v = \langle 0, 0, u\rangle$. So flux $= \int_0^{\frac{\pi}{2}}\int_0^3 \langle 0, 0, u\cos v\rangle \cdot \langle 0, 0, u\rangle\, dudv = 9$.

3. (**10 points**) Compute the acceleration of the velocity field $\mathbf{v} = \langle\frac{y}{\sqrt{x+y}}, -\frac{x}{\sqrt{x-y}}\rangle$.

   solution: $\mathbf{a} = \langle\frac{-y^2}{2\,(x+y)^2} - \frac{x\,(2\,x+y)}{2\,\sqrt{x-y}\,(x+y)^{\frac{3}{2}}}, \frac{x^2}{2(x-y)^2} - \frac{(x-2\,y)\,y}{2(x-y)^{\frac{3}{2}}\,\sqrt{x+y}}, 0\rangle$.

4. (**30 Points**) Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ form a basis of unit vectors at a typical point on Earth, which rotates with angular velocity $\mathbf{\Omega}$. Let $\mathbf{v} = u\mathbf{e}_1 + v\mathbf{e}_2 + w\mathbf{e}_3$ be a typical velocity vector measured at a typical point P on Earth.

   (a) Show that $\mathbf{f} = 2\mathbf{\Omega} \times \mathbf{v}$ measures the Coriolis force imparted on the fluid particle P.

   Solution: The absolute velocity of a particle is the sum of $\mathbf{v}$, its relative velocity, and $\mathbf{\Omega} \times \mathbf{r}$, the velocity imparted on the particle by the rotation of the planet. Since $\mathbf{v} = \frac{d\mathbf{r}}{dt}$, we define the total time derivative of position $\mathbf{r}$ by

$$\frac{D}{Dt} = \frac{d}{dt} + \mathbf{\Omega} \times .$$

   (so that $\frac{D\mathbf{r}}{Dt} = \frac{d\mathbf{r}}{dt} + \mathbf{\Omega} \times \mathbf{r}$.) By definition, absolute acceleration is $\frac{D^2\mathbf{r}}{Dt^2}$ or

$$\frac{D^2\mathbf{r}}{Dt^2} = (\frac{d}{dt} + \mathbf{\Omega}\times)(\frac{d}{dt} + \mathbf{\Omega}\times)\mathbf{r}$$

which equals

$$(\frac{d}{dt} + \mathbf{\Omega}\times)(\frac{d\mathbf{r}}{dt} + \mathbf{\Omega} \times \mathbf{r})$$

which equals

$$\frac{d^2\mathbf{r}}{dt^2} + 2\mathbf{\Omega} \times \mathbf{r} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}).$$

The first term is the relative acceleration (what we meaure on the planet), the second is Coriolis, and the third is the centripetal acceleration.

(b) Find the components of $\mathbf{f}$ in terms of Ω, the magnitude of $\mathbf{\Omega}$, and ϕ the latitude of P.

Solution: Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis in spherical coordinates at a typical point on the planet. Let ϕ be the latitude of that point. Then $\mathbf{\Omega} = \langle 0, \Omega\cos\phi, \Omega\sin\phi\rangle$ and $\mathbf{v} = \langle u, v, w\rangle$ in that basis (note that $\mathbf{\Omega}$ and $\mathbf{v}$ will have very different components in the cartesian basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. And $\mathbf{f} = 2\langle \Omega\, w\, \cos\phi - \Omega\, v\, \sin\phi, \Omega\, u\, \sin\phi, -\Omega\, u\, \cos\phi\rangle$. In typical mesoscale modeling we ignore w and the third component of $\mathbf{f}$. With the notation $f = 2\Omega\sin\phi$, $\mathbf{f}$ becomes

$$\mathbf{f} = \langle -fv, fu, 0\rangle.$$

(c) Compute $\mathbf{f}$ when P is at the north pole; When P is at the equator.

Solution: $\phi = \frac{\pi}{2}$ at the north pole so $f = 2\Omega$ so $\mathbf{f} = \langle 2\Omega v, 2\Omega u\rangle$. What is $\mathbf{e}_1$ at the north pole? At the equator $\phi = 0$ so $f = 0$ and thus $\mathbf{f} = \mathbf{0}$.

5. (**20 Points**) A flow is called geostrophic if the velocity field $\mathbf{v}$ and pressure gradient ∇p satisfy

$$\rho f v_2 = \frac{\partial p}{\partial x}, \qquad \rho f v_1 = -\frac{\partial p}{\partial y},$$

where $f = 2\Omega\sin\phi$. Assume the velocity field is incompressible. Let ψ be the stream function of this flow. Prove that the isobars of the flow and particle paths of the flow must coincide. (Hint: Use your knowledge of gradient of a function and its relation to the level curves of that function).

Solution: We know that ∇p is perpendicular to its isobar. Let P be a point on the isobar. To show that the velocity at P is tangential to the isobar, we will show that ∇p is perpendicular to $\mathbf{v}$. But from the geostrophic equation we have $v_1 = \frac{1}{\rho f}\frac{\partial p}{\partial y}$ and $v_2 = -\frac{1}{\rho f}\frac{\partial p}{\partial x}$. So

$$\mathbf{v} \cdot \nabla p = \langle v_1, v_2\rangle \cdot \langle \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}\rangle = \langle \frac{1}{\rho f}\frac{\partial p}{\partial y}, -\frac{1}{\rho f}\frac{\partial p}{\partial x}\rangle \cdot \langle \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}\rangle = 0$$

What is the relation between particle paths and streamlines?