SM311o – Third Exam, 8 Dec, 1998

1. Let $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ be the standard basis for the three dimensional space. Let $\mathbf{a}(t) = a_1(t)\mathbf{i} + a_2(t)\mathbf{j} + a_3(t)\mathbf{k}$ and $\mathbf{b}(t) = b_1(t)\mathbf{i} + b_2(t)\mathbf{j} + b_3(t)\mathbf{k}$. Prove that

$$\frac{d}{dt}\big(\mathbf{a} \times \mathbf{b}\big) = \mathbf{a}' \times \mathbf{b} + \mathbf{a} \times \mathbf{b}'.$$

2. Let $\mathbf{\Omega}$ be the angular velocity of the Earth. Let $\mathbf{v}$ be the relative velocity of a fluid particle located at P with latitude ϕ. Let unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be a basis at P with $\mathbf{e}_1$ pointing east, $\mathbf{e}_2$ pointing north, and $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$.

   (a) Draw a diagram that shows the relationship between $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{\Omega}$.

   (b) Show that $\mathbf{\Omega} = \Omega \cos\phi \mathbf{e}_2 + \Omega \sin\phi \mathbf{e}_3$.

   (c) Let $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$. Find the three components of

$$\mathbf{\Omega} \times \mathbf{v}.$$

3. Consider the Geostrophic equations

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + A_V\frac{\partial^2 u}{\partial z^2}, \quad fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + A_V\frac{\partial^2 v}{\partial z^2}, \quad g = -\frac{1}{\rho}\frac{\partial p}{\partial z}.$$

   Suppose that u and v are functions of z only and that ρ, f, and A_V are constants.

   (a) Show that the horizontal pressure gradient $\langle \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y} \rangle$ must be constant.

   (b) Suppose that $u = U$, a constant, and $v = 0$ when $z \to \infty$. Find the constant values of the horizontal pressure gradient.

4. As we have seen in class, the horizontal component of of velocity u of the previous problem satisfies the 4-th order ODE

$$u'''' + \alpha^2 u = 0.$$

   (a) Start by looking for solutions of this equation in the form $u(z) = e^{mz}$. Find the polynomial that m must satify.

   (b) Show by direct computation that $(\pm\frac{1}{\sqrt{2}}(1+i))^2 = i$ and $(\pm\frac{1}{\sqrt{2}}(1-i))^2 = -i$. Use these facts to write down the general solution of the 4-th order problem.