SM311o – Third Exam, 8 Dec, 1998 (Solutions)

1. Let $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ be the standard basis for the three dimensional space. Let $\mathbf{a}(t) = a_1(t)\mathbf{i} + a_2(t)\mathbf{j} + a_3(t)\mathbf{k}$ and $\mathbf{b}(t) = b_1(t)\mathbf{i} + b_2(t)\mathbf{j} + b_3(t)\mathbf{k}$. Prove that

$$\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \mathbf{a}' \times \mathbf{b} + \mathbf{a} \times \mathbf{b}'.$$

Solution: Note that

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - b_1a_2 \rangle.$$

So

$$\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \langle a_2'b_3 + a_2b_3' - a_3'b_2 - a_3b_2', a_3'b_1 + a_3b_1' - a_1'b_3 - a_1b_3', a_1'b_2 + a_1b_2' - b_1'a_2 - b_1a_2' \rangle$$

which can be separated into the two vectors

$$\langle a_2'b_3 - a_3'b_2, a_3'b_1 - a_1'b_3, a_1'b_2 - b_1'a_2 \rangle + \langle a_2b_3' - a_3b_2', a_3b_1' - a_1b_3', a_1b_2' - b_1'a_2 \rangle$$

which in turn is equal to $\mathbf{a}' \times \mathbf{b} + \mathbf{a} \times \mathbf{b}'$.

2. Let $\mathbf{\Omega}$ be the angular velocity of the Earth. Let $\mathbf{v}$ be the relative velocity of a fluid particle located at P with latitude ϕ. Let unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be a basis at P with $\mathbf{e}_1$ pointing east, $\mathbf{e}_2$ pointing north, and $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$.

   (a) Draw a diagram that shows the relationship between $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{\Omega}$.

   (b) Show that $\mathbf{\Omega} = \Omega \cos\phi \mathbf{e}_2 + \Omega \sin\phi \mathbf{e}_3$.

   (c) Let $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$. Find the three components of

$$\mathbf{\Omega} \times \mathbf{v}.$$

Solution: a) See Figure 1.

b) The three vectors $\mathbf{\Omega}$, $\mathbf{e}_2$, $\mathbf{e}_3$ form a right triangle as shown in Figure 1. Thus, the projection of $\mathbf{\Omega}$ on $\mathbf{e}_2$ is $\Omega \cos\phi \mathbf{e}_2$. Similarly, the projrction of $\mathbf{\Omega}$ on $\mathbf{e}_3$ is $\Omega \sin\phi \mathbf{e}_3$.

c)

$$\mathbf{\Omega} \times \mathbf{v} = (\Omega \cos\phi \mathbf{e}_2 + \Omega \sin\phi \mathbf{e}_3) \times (v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3)$$

which in turn is equal to

$$\Omega v_1 \cos\phi \mathbf{e}_2 \times \mathbf{e}_1 + \Omega v_3 \cos\phi \mathbf{e}_2 \times \mathbf{e}_3 + \Omega v_1 \sin\phi \mathbf{e}_3 \times \mathbf{e}_1 + \Omega v_2 \sin\phi \mathbf{e}_3 \times \mathbf{e}_2.$$

But $\mathbf{e_2} \times \mathbf{e}_1 = -\mathbf{e}_3$, $\mathbf{e_2} \times \mathbf{e}_3 = \mathbf{e}_1$, $\mathbf{e_3} \times \mathbf{e}_1 = \mathbf{e}_2$, and $\mathbf{e_3} \times \mathbf{e}_2 = -\mathbf{e}_1$ so

$$\mathbf{\Omega} \times \mathbf{v} = -v_1\Omega \cos\phi \mathbf{e}_3 + v_3 \cos\phi \mathbf{e}_1 + v_1\Omega \sin\phi \mathbf{e}_2 - v_2\Omega \mathbf{e}_1.$$

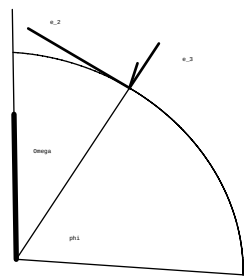

Figure 1: The relationship between $\mathbf{\Omega}$, $\mathbf{e}_2$, $\mathbf{e}_3$.

3. Consider the viscous Geostrophic equations

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + A_V \frac{\partial^2 u}{\partial z^2}, \quad fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + A_V \frac{\partial^2 v}{\partial z^2}, \quad g = -\frac{1}{\rho}\frac{\partial p}{\partial z}.$$

Suppose that u and v are functions of z only and that ρ, f, and A_V are constants.

(a) Show that the horizontal pressure gradient $\langle \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y} \rangle$ must be constant.

Solution: Differentiating the third equation with respect to x and y yields $\frac{\partial^2 p}{\partial x \partial z} = \frac{\partial^2 p}{\partial y \partial z} = 0$. Therefore, $\frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial y}$ are independent of z. Now differentiate the first equation in the Geostrophic system with respect to x. Since u and v only depend on z and ρ and f are constants, we have $\frac{\partial^2 p}{\partial x^2} = 0$, which implies that $\frac{\partial p}{\partial x}$ is independent of x. Similarly, differentiating the first equation with respect to y shows that $\frac{\partial^2 p}{\partial x \partial y} = 0$, which implies that $\frac{\partial p}{\partial x}$ is independent of y. Putting all of the information about $\frac{\partial p}{\partial x}$ together, we see that this term must be contant. A similar argument applied to the second equation of the Geostrophic system shows that $\frac{\partial p}{\partial y}$ must be constant.

(b) Suppose that $u = U$, a constant, and $v = 0$ when $z \to \infty$. Find the constant values of the horizontal pressure gradient.

Solution: Since u and v have asymptotic limits as $z \to \infty$, the terms $u''(z)$ and $v''(z)$ are zero as $z \to \infty$ (draw a picture of such a function to convince yourself of this fact). Now let z approach infinity in the first equation in the Geostrophic system. Since $\lim_{z\to\infty} v = 0$, we have $\lim_{z\to\infty} \frac{\partial p}{\partial x} = 0$. But we proved in the previous problem that $\frac{\partial p}{\partial x}$ is constant. Therefore, $\frac{\partial p}{\partial x} = 0$ everywhere. Similarly, let $z \to \infty$ in the second equation of the Geostrophic system. We get

$$\rho f U = -\frac{\partial p}{\partial y},$$

which is the value of $\frac{\partial p}{\partial y}$ everywhere since this term is constant.

4. As we have seen in class, the horizontal component of of velocity u of the previous problem satisfies the 4-th order ODE

$$u'''' + \alpha^2 u = 0.$$

(a) Start by looking for solutions of this equation in the form $u(z) = e^{mz}$. Find the polynomial that m must satify.

Solution: $m^4 + \alpha^2 = 0$.

(b) Show by direct computation that $(\pm\frac{1}{\sqrt{2}}(1+i))^2 = i$ and $(\pm\frac{1}{\sqrt{2}}(1-i))^2 = -i$. Use these facts to write down the general solution of the 4-th order problem.

Solution: The first assertions are straightforward. Let $\gamma = \sqrt{\frac{\alpha}{2}}$. Then $m = \gamma + \gamma i$ is a root of the auxiliary polynomial. Therefore, $u(z) = e^{(\gamma+\gamma i)z}$ is a solution of $u'''' + \alpha^2 u = 0$. But

$$e^{(\gamma+\gamma i)z} = e^{\gamma z}e^{\gamma i z} = e^{\gamma z}(\cos\gamma z + i\sin\gamma z),$$

from which we construct the real-valued solution

$$c_1 e^{\gamma z}\cos\gamma z + c_2 e^{\gamma z}\sin\gamma z.$$

Similarly, $m = -\gamma - \gamma i$ gives rise to the real-valued solution

$$c_3 e^{-\gamma z}\cos\gamma z + c_4 e^{-\gamma z}\sin\gamma z.$$

Thus, the general solution of $u'''' + \alpha^2 u = 0$ is

$$u(z) = c_1 e^{\gamma z}\cos\gamma z + c_2 e^{\gamma z}\sin\gamma z + c_3 e^{-\gamma z}\cos\gamma z + c_4 e^{-\gamma z}\sin\gamma z.$$