SM311O (Third Exam) **Solutions**

21 April 1999

1. (**20 Points**) Give a parametrization of the following surfaces:

   (a) The plane that passes though the points $(1,0,0)$, $(0,2,0)$ and $(0,1,1)$.
   Solution: The equation of any plane is of the form $Ax + By + Cz = D$. Substituting the point $(1,0,0)$ into this equation yields $A = D$. Similarly, substituting $(0,2,0)$ and $(0,1,1)$ into $Ax + By + CZ = D$ yields $2B = D$ and $B + C = D$. So $B = \frac{D}{2}$ and $C = \frac{D}{2}$. Substituting the values of A, B, and C in terms of D into $Ax + By + Cz = D$ and dividing by D yields

   $$x + \frac{y}{2} + \frac{z}{2} = 1$$

   for the equation of the plane. Note that this equation can alternatively be written as

   $$z = 2 - 2x - y.$$

   Hence, the plane can now be parametrized as

   $$\mathbf{r}(u,v) = \langle u, v, 2 - 2u - v\rangle.$$

   (b) The disk of radius 2 located in the yz-plane and centered at $(0,2,3)$.
   Solution: The equation for the circle that constitutes the boundary of this region is given by $x = 0$ and $(y-2)^2 + (z-3)^2 = 4$. So start the parametrization by letting $x = 0$, $y - 2 = u\cos v$ and $z - 3 = u \sin v$ where $0 < u < 2$ and $0 < v < 2\pi$. The parametrization is then

   $$\mathbf{r}(u,v) = \langle 0, 2 + u\cos v, 3 + u \sin v\rangle.$$

2. (**15 Points**) Use double integration to determine the volume of the tetrahedron with vertices $(0,0,0)$, $(2,0,0)$, $(0,3,0)$, and $(0,0,4)$.

   Solution: Following the same procedure as the one described in 1(a), the equation of the plane that passes through the points $(2,0,0)$, $0,3,0)$ and $0,0,4)$ is

   $$\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$$

from which we get that $z = 4 - 2x - \frac{4y}{3}$ is the equation of the slanted face of the tetraderon. Hence, the volume of the tetrahedron is

$$\int_0^3 \int_0^{2-\frac{2y}{3}} \left(4 - 2x - \frac{4y}{3}\right) dxdy = 4.$$

3. (**15 Points**) Compute the flux of the velocity field $\mathbf{v} = (y - y^2)\mathbf{k}$ through the surface S, the disk of radius 3 in the xy-plane centered at the origin.

   Solution: To compute the flux, we need to determine $\int\int_D \mathbf{v} \cdot \mathbf{r}_u \times \mathbf{r}_v \, dudv$, where $\mathbf{r}$ is the parametrization of the surface. So first we must parametrize the disk of radius 3 in the xy-plane centered at the origin. Following the strategy in 1(b), we have

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, 0 \rangle, \quad 0 < u < 3, \quad 0 < v < 2\pi.$$

   Now

$$\mathbf{r}_u \times \mathbf{r}_v = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 0 \end{bmatrix} = \langle 0, 0, u \rangle.$$

   So

$$\int\int_D \mathbf{v} \cdot \mathbf{r}_u \times \mathbf{r}_v \, dudv = \int_0^3 \int_0^{2\pi} \langle 0, 0, u \sin v - u^2 \sin^2 v \rangle \cdot \langle 0, 0, u \rangle \, dvdu =$$

$$\int_0^3 \int_0^{2\pi} \left(u^2 \sin v - u^3 \sin^2 v\right) dvdu = -\frac{81\pi}{4}.$$

4. (**20 Points**) Compute the flux of the vorticity for the velocity field $\mathbf{v} = \langle y, -x, z \rangle$ through the surface of the hemisphere $x^2 + y^2 + z^2 < 1$ and $z \geq 0$. (Hint: You may calculate the flux directly, or indirectly by applying the Stokes theorem, which states that the flux of vorticity along a surface S is equal to the flow's circulation around the boundary of S.)

   Solution: I'll compute the flux of the vorticity indirectly, using the Stokes Theorem, which states that

$$\int\int_D \nabla \times \mathbf{v} \cdot d\mathbf{A} = \oint_C \mathbf{v} \cdot d\mathbf{r}.$$

   So, to compute $\oint_C \mathbf{v} \cdot d\mathbf{r}$ we must first parametrize the curve C, which is a circle of radius 1 located in the xy-plane and centered at the origin:

$$\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle.$$

Now

$$\oint_C \mathbf{v}\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{v}\cdot\frac{d\mathbf{r}}{dt}\,dt = \int_0^{2\pi} \langle \sin t, -\cos t, 0\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dt =$$

$$\int_0^{2\pi} -(\sin^2 t + \cos^2)\, dt = -2\pi.$$

5. (**30 Points**) Let $\mathbf{v} = \langle \frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2}, 0\rangle$.

   (a) Determine the acceleration of $\mathbf{v}$.
   Solution: $\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}$. Now

$$\nabla\mathbf{v} = \begin{bmatrix} \frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y} \\ \frac{\partial v_2}{\partial x} & \frac{\partial v_2}{\partial y} \end{bmatrix} = \begin{bmatrix} -\frac{2xy}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2} \\ \frac{x^2-y^2}{(x^2+y^2)^2} & \frac{2xy}{(x^2+y^2)^2} \end{bmatrix}.$$

   Now, $\frac{\partial \mathbf{v}}{\partial t} = \mathbf{0}$ (why?) while $\mathbf{v}\cdot\nabla\mathbf{v}$ means $(\nabla\mathbf{v})\mathbf{v}$ in the sense of matrix multiplication. So

$$\mathbf{a} = \begin{bmatrix} -\frac{2xy}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2} \\ \frac{x^2-y^2}{(x^2+y^2)^2} & \frac{2xy}{(x^2+y^2)^2} \end{bmatrix} \begin{bmatrix} \frac{y}{x^2+y^2} \\ -\frac{x}{x^2+y^2} \end{bmatrix} = \begin{bmatrix} -\frac{x}{(x^2+y^2)^2} \\ -\frac{y}{(x^2+y^2)^2} \end{bmatrix} =$$

$$-\langle \frac{x}{(x^2+y^2)^2}, \frac{y}{(x^2+y^2)^2}\rangle.$$

   (b) Determine if the above acceleration has a potential. If it does, find it.
   Solution: For $\mathbf{a}$ to have a potential it must satisfy $\nabla\times\mathbf{a} = \mathbf{0}$ which the acceleration in 5(a) does satisfy. Then there must exist a function ϕ such that

$$\frac{\partial\phi}{\partial x} = -\frac{x}{(x^2+y^2)^2}, \quad \frac{\partial\phi}{\partial y} = -\frac{y}{(x^2+y^2)^2}. \tag{1}$$

   Integrating the first equation with respect to x yields

$$\phi = \frac{1}{2(x^2+y^2)} + f(y).$$

   Differentiating this equation with respect to y and comparing the result with the second equation in (1) shows that $f'(y) = 0$, or $f = \text{const}$. Let this constant be zero. Then $\phi = \frac{1}{2(x^2+y^2)}$.