SM3110, Third Exam
28 April 2000

1) Parametrizations:

a) Straight line connecting (1,2,3) to (1,4,9): $m = \frac{9-3}{4-2} = 3$

$$\underset{\sim}{r}(t) = \langle 1, t, 3t-3 \rangle.$$

b) Disk of radius 2 centered at the origin and located in the plane $z=1$.

$$\underset{\sim}{r}(u,v) = \langle u\cos v,\ u\sin v,\ 1\rangle, \quad \begin{matrix} 0 < u < 2 \\ 0 < v < 2\pi. \end{matrix}$$

c) Circle of radius 2 centered at (1,1,1), located in the plane $z=1$.

$$\underset{\sim}{r}(t) = \langle 1+2\cos t,\ 1+2\sin t,\ 1\rangle, \quad 0 < t < 2\pi.$$

d) $z = 1 - x^2 - y^2$, domain $x^2+y^2 \le 1$.

$$\underset{\sim}{r}(u,v) = \langle u\cos v,\ u\sin v,\ 1-u^2\rangle$$

2) Flux of $\underset{\sim}{V} = x^2 \underset{\sim}{k}$, S disk of radius 3 centered at origin, located in xy-plane.

$$\underset{\sim}{r}(u,v) = \langle u\cos v,\ u\sin v,\ 0\rangle, \quad 0<u<3, \quad 0<v<2\pi.$$

$$\underset{\sim}{r}_u \times \underset{\sim}{r}_v = u\,\underset{\sim}{k}.$$

$$\text{Flux} = \iint_S \underset{\sim}{V}\cdot d\underset{\sim}{A} = \int_0^3\int_0^{2\pi} \langle 0,0,u^2\cos^2 v\rangle \cdot \langle 0,0,u\rangle\, dv\, du$$

$$= \frac{81\pi}{4} \quad (\text{from Mathematica})$$

3) Acceleration of $\underset{\sim}{V} = \left\langle \frac{y}{\sqrt{x^2+y^2}},\ -\frac{x}{\sqrt{x^2+y^2}} \right\rangle$

$$a_1 = \frac{d}{dt}\frac{y}{\sqrt{x^2+y^2}} = \frac{d}{dt}\left(y(x^2+y^2)^{-1/2}\right) = \frac{dy}{dt}(x^2+y^2)^{-1/2} + y\left(-\tfrac{1}{2}\right)(x^2+y^2)^{-3/2}\left(2x\frac{dx}{dt}+2y\frac{dy}{dt}\right)$$

$$= \frac{-x}{\sqrt{x^2+y^2}}\,\frac{1}{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{(x^2+y^2)^3}}\left(\frac{xy}{\sqrt{x^2+y^2}} - \frac{yx}{\sqrt{x^2+y^2}}\right) = \frac{-x}{x^2+y^2}$$

Similarly,

$$a_2 = \frac{-y}{x^2+y^2}. \quad \text{So} \quad \underset{\sim}{a} = -\frac{1}{x^2+y^2}\langle x, y\rangle$$

4) a) $\Omega = 2\pi(\text{radians}) / (24 \text{ hours})(60 \text{ minutes/hour})(60 \text{ seconds/minute})$

$= 0.00007$ rad/sec.

b) $\|\underset{\sim}{\Omega} \times \underset{\sim}{r}\| = \|\underset{\sim}{\Omega}\| \, \|\underset{\sim}{r}\| \sin\alpha$, where α is the angle between $\underset{\sim}{\Omega}$ and $\underset{\sim}{r}$.

Choose $\underset{\sim}{r}$ on the equator. Then $\alpha = 90°$ and $\|\underset{\sim}{r}\| = 6000$ kilometers.

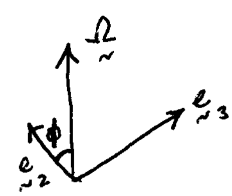
$\underset{\sim}{\Omega} \times \underset{\sim}{r}$ will be perpendicular to $\underset{\sim}{\Omega}$ so

$$\|\underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \underset{\sim}{r})\| = \|\underset{\sim}{\Omega}\| \, \|\underset{\sim}{\Omega} \times \underset{\sim}{r}\| = \|\underset{\sim}{\Omega}\|^2 \|\underset{\sim}{r}\|$$

$$= (.00007)^2 \times 6000 \text{ kilometer} \times 1000 \text{ meters/kilometer}$$

$$= 0.03 \text{ m/sec}^2$$

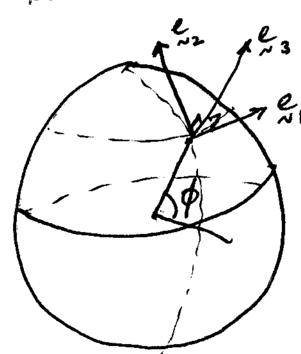
5) In the $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ basis vector $\underset{\sim}{\Omega}$ is



$$\underset{\sim}{\Omega} = \Omega \cos\phi \, \underset{\sim}{e}_2 + \Omega \sin\phi \, \underset{\sim}{e}_3$$

Then $2\underset{\sim}{\Omega} \times \underset{\sim}{v} = (2\Omega\cos\phi\, \underset{\sim}{e}_2 + 2\Omega\sin\phi\, \underset{\sim}{e}_3)$

$\times (u\underset{\sim}{e}_1 + v\underset{\sim}{e}_2 + w\underset{\sim}{e}_3)$



$= 2\Omega\cos\phi\, u\, \underset{\sim}{e}_2 \times \underset{\sim}{e}_1 + 2\Omega\cos\phi\, w\, \underset{\sim}{e}_2 \times \underset{\sim}{e}_3$

$+ 2\Omega\sin\phi\, u\, \underset{\sim}{e}_3 \times \underset{\sim}{e}_1 + 2\Omega v \sin\phi\, \underset{\sim}{e}_3 \times \underset{\sim}{e}_2$

But $\underset{\sim}{e}_2 \times \underset{\sim}{e}_1 = -\underset{\sim}{e}_3$ (see figure)

$\underset{\sim}{e}_2 \times \underset{\sim}{e}_3 = \underset{\sim}{e}_1$, $\underset{\sim}{e}_3 \times \underset{\sim}{e}_1 = \underset{\sim}{e}_2$, $\underset{\sim}{e}_3 \times \underset{\sim}{e}_2 = -\underset{\sim}{e}_1$

So, $2\underset{\sim}{\Omega} \times \underset{\sim}{v} = \langle \underbrace{-2\Omega\sin\phi\, v + 2\Omega\cos\phi\, W}_{\text{component in } \underset{\sim}{e}_1 \text{ direction}},\ 2\Omega\sin\phi\, u,\ -2\Omega\cos\phi\, u \rangle$

North pole: $\phi = \frac{\pi}{2} \Rightarrow \underset{\sim}{f} = \langle -2\Omega v, 2\Omega u, 0 \rangle$

Equator: $\phi = 0 \Rightarrow \underset{\sim}{f} = \langle 2\Omega w, 0, -2\Omega u \rangle \approx \underset{\sim}{0}$.

6) $y'' - iy = 0$. $y(x) = e^{mx}$. $\Rightarrow m^2 - i = 0 \Rightarrow$

$m = \pm\sqrt{i} = \pm \frac{1}{\sqrt{2}}(1+i) \Rightarrow y(x) = e^{\frac{1}{\sqrt{2}}(1+i)x} = e^{\frac{x}{\sqrt{2}}} e^{i\frac{x}{\sqrt{2}}}$

$= e^{\frac{x}{\sqrt{2}}} \left(\cos\frac{x}{\sqrt{2}} + i \sin\frac{x}{\sqrt{2}} \right)$

$\Rightarrow$ General solution $\boxed{y(x) = c_1 e^{\frac{x}{\sqrt{2}}} \cos\frac{x}{\sqrt{2}} + c_2 e^{\frac{x}{\sqrt{2}}} \sin\frac{x}{\sqrt{2}}}$.