Final Examination, 15 December 1998
SM311O (Fall 1998)

The following formulas may be useful to you:

$$a)\ \oint_C \mathbf{v}\cdot d\mathbf{r} = \int\int_S \nabla\times\mathbf{v}\cdot d\mathbf{A},$$

$$b)\ \ \rho(\frac{\partial\mathbf{v}}{\partial t}+\nabla\mathbf{v}\cdot\mathbf{v}) = -\nabla p+\mu\Delta\mathbf{v}+\rho\mathbf{F}, \qquad \text{div } \mathbf{v}=0.$$

$$c)\quad -fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}+A_V\frac{\partial^2 u}{\partial z^2},\quad fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}+A_V\frac{\partial^2 v}{\partial z^2},\quad 0=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g.$$

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1. (a) Let $\mathbf{v}=\langle x^2z, y\sin(xy), e^{-xz}\rangle$. Find the divergence of $\mathbf{v}$.
   Solution: div $(\langle x^2y, \sin(xy), e^{-xz}\rangle = \frac{\partial(x^2y)}{\partial x}+\frac{\partial\sin xy)}{\partial y}+\frac{\partial e^{-xz}}{\partial z}$ $=2xy+x\cos y-xe^{-xz}$.

   (b) $\mathbf{v}=\langle\frac{y}{\sqrt{x^2+y^2}}, -\frac{x}{\sqrt{x^2+y^2}}, 0\rangle$. Find the curl of $\mathbf{v}$.
   Solution: $\nabla\times\langle\frac{y}{\sqrt{x^2+y^2}}, -\frac{x}{\sqrt{x^2+y^2}}, 0\rangle = \langle 0,0,-\frac{1}{\sqrt{x^2+y^2}}\rangle$.

   (c) Let $f(x,y)=e^{2x}\sin 3\pi y$. Find the direction of steepest descent at $P=(0,\frac{1}{4})$.
   Solution: $\nabla f = \langle 2e^{2x}\sin 3\pi y, 3\pi e^{2x}\cos 3\pi y\rangle$. So the direction of steepest descent at $P=(0,\frac{1}{4})$ is

$$\nabla f|_P = \langle 2\sin\frac{3\pi}{4}, 3\pi\cos\frac{3\pi}{4}\rangle = \langle\sqrt{2}, -\frac{3\pi\sqrt{2}}{2}\rangle.$$

2. Verify by direct differentiation if

   (a) $u(z)=e^z\cos z$ is a solution of $u''''+a^2u=0$ for any a.
   Solution: $u''''(z)=-4e^z\cos z$. So $u''''+a^2u=0$ if $a^2=4$ or if $a=\pm 2$.

   (b) $u(x,y)=\sin 3x\sin 4y$ is an eigenfunction of the Laplace operator $-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}$. What is the eigenvalue?
   Solution: Let $u(x,y)=\sin 3x\sin 4y$. Then

$$-(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}) = 25\sin 3x\sin 4y.$$

   So $\lambda=25$.

3. Parametrize the following curves and surfaces:

(a) The plane passing through the points $(1,1,0)$, $(0,3,2)$, and $(1,0,5)$.
Solution: The equation of the plane passing through $(1,1,0)$, $(0,3,2)$, and $(1,0,5)$ is of the form $Ax+By+Cz=D$. Substituting the three points into this equation leads to the following system of algebraic equations

$$A+B=D, \quad 3B+2C=D, \quad x+5C=D$$

whose solution is $A=\frac{12D}{17}$, $B=\frac{5D}{17}$, $C=\frac{D}{17}$. Substituting these values in $Ax+By+Cz=D$ and dividing by D yields $12x+5y+z=17$ for the equation of the plane, or equivalently, $z=17-12x-5y$. So

$$\mathbf{r}(u,v)=\langle u,v,17-12u-5v\rangle$$

is the parametrization of the plane.

(b) The curve $x^2+3y^2=2$.
Solution: Write the equation of the curve as $\frac{x^2}{2}+\frac{3y^2}{2}=1$. Let $\cos^2 t=\frac{x^2}{2}$ and $\sin^2 t=\frac{3y^2}{2}$ or $x=\sqrt{2}\cos t$ and $y=\sqrt{\frac{2}{3}}\sin t$. Then

$$\mathbf{r}(t)=\langle\sqrt{2}\cos t,\sqrt{\frac{2}{3}}\sin t\rangle, \quad t\in(0,2\pi)$$

is the parametrization of this curve.

(c) The upper hemisphere of radius 2 centered at $(1,-1,1)$.
Solution: $\mathbf{r}(u,v)=\langle 1+2\cos u\sin v,-1+2\sin u\sin v,1+2\cos v\rangle, \quad u\in(0,2\pi), \quad v\in(0,\frac{\pi}{2})$.

4. (a) The function $\phi(x,y,z)=ax^2y^2+by^2z^2+cz^2x^2-x$ is the potential for a velocity vector field $\mathbf{v}$. Determine the values of a, b, and c so that the velocity of the particle located at $(1,-2,3)$ is zero.
Solution: Because $\mathbf{v}=\nabla\phi$ we have

$$\mathbf{v}=\langle 2axy^2+2cxz^2-1,2ax^2y+2byz^2,2by^2z+2cx^2z\rangle.$$

Substitute $(1,-2,3)$ into $\mathbf{v}$:

$$\mathbf{v}|_{(1,-2,3)}=\langle -1+8a+18c,-4a-36b,24b+6c\rangle$$

which is $\mathbf{0}$ if $a=\frac{1}{16}$, $b=-\frac{1}{144}$, and $c=\frac{1}{36}$.

(b) The function $\psi(x,y)=ax^2-xy+by^2$ is the stream function of a velocity field $\mathbf{v}$. Find a and b so that the velocity of the particle located at $(-1,4)$ is perpendicular to $\langle 1,1\rangle$.
Solution: Since $\psi=ax^2-xy+by^2$, then

$$\mathbf{v}=\langle -x+2by,-2ax+y\rangle.$$

Now, the velocity at $(-1,4)$ is $\mathbf{v}|_{(-1,4)}=\langle 1+8b,2a+4\rangle$. After setting the dot product of this vector with $\langle 1,1\rangle$ equal to zero we get that $2a+8b=-5$.

5. (a) Consider the velocity vector field $\mathbf{v} = \langle x^2 - y^2, -2xy + x^3 \rangle$. Does this vector field have a stream function? If yes, find it.

   Solution: div $\mathbf{v} = 2x - 2x = 0$, so the velocity field have a stream function. From $\frac{\partial \psi}{\partial y} = x^2 - y^2$ we have $\psi = x^2 y - \frac{y^3}{3} + f(x)$. But $\frac{\partial \psi}{\partial x} = 2xy - x^3$ while the same derivative of ψ also equals $2xy + f'(x)$. So $f'(x) = -x^3$ or $f(x) = -\frac{x^4}{4}$. Finally,

$$\psi = x^2 y - \frac{y^3}{3} - \frac{x^4}{4}.$$

   (b) Let u and v be the velocities in a geostrophic flow with $A_V = 0$. What is the geometric relationship between the isobars and the particle paths of typical fluid particles?

   Solution: According to the viscous geostrophic equations, we have

$$v = \frac{1}{\rho f}\frac{\partial p}{\partial x}, \quad u = -\frac{1}{\rho f}\frac{\partial p}{\partial y},$$

   when $A_V = 0$. Thus $\mathbf{v} \cdot \nabla p = \langle u, v \rangle \cdot \langle \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y} \rangle = \langle -\frac{1}{\rho f}\frac{\partial p}{\partial y}, \frac{1}{\rho f}\frac{\partial p}{\partial x} \rangle \cdot \langle \langle \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y} \rangle = 0$. Therefore, the velocity field is orthogonal to the pressure gradient in the flow. Since ∇p is orthogonal to the isobars, $\mathbf{v}$ is parallel with isobars which implies that the isobars and the particle paths coincide.

6. A flow is called barotropic if the pressure field p is a function of the density ρ, that is, $p = f(\rho)$ for some function f.

   (a) Compute ∇p in terms of $\nabla \rho$. What can you conclude about $\nabla p \times \nabla \rho$ in a barotropic flow? Why? What does this result say about the isobars and isopycnals of the flow? Why?

   Solution: $\nabla p = \langle \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \rangle = \langle f'(\rho)\frac{\partial \rho}{\partial x}, f'(\rho)\frac{\partial \rho}{\partial y}, f'(\rho)\frac{\partial \rho}{\partial z} \rangle = f'(\rho)\nabla \rho$. So

$$\nabla p = f'(\rho)\nabla \rho.$$

   Hence, ∇p and $\nabla \rho$ are parallel which implies that $\nabla p \times \nabla \rho = \mathbf{0}$. Also, because ∇p and $\nabla \rho$ are perpendicular to isobars and isopycnals, respectively, then isobars and isopycnals are parallel.

   (b) Let $f(x) = x^2 + 3x$. Suppose that the density at $P = (1, 2, 3)$ is 1.003. Furthermore, suppose that the pressure gradient at P is $\langle 3, 4, 2 \rangle$. What is the density gradient at P?

   Solution: Since $\nabla p = f'(\rho)\nabla \rho$, then $\nabla \rho = \frac{1}{f'(\rho)}\nabla p$. But

$$f'(x) = 2x + 3$$

   so $f'(1.003) = 5.006$. Also, since $\nabla p|_{(1,2,3)} = \langle 3, 4, 2 \rangle$, we have

$$\nabla \rho|_{(1,2,3)} = 5.006\langle 3, 4, 2 \rangle = \langle 15.018, 20.024, 10.012 \rangle.$$

7. (a) Let p and ρ be two arbitrary smooth functions of x, y, and z. Use direct differentiation and prove the identity

$$\nabla \times \left(\frac{1}{\rho}\nabla p\right) = -\frac{1}{\rho^2}(\nabla\rho \times \nabla p).$$

Solution: Let's start by expanding the left-hand side of this identity:

$$\nabla \times \left(\frac{1}{\rho}\nabla p\right) = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{\rho}\frac{\partial p}{\partial x} & \frac{1}{\rho}\frac{\partial p}{\partial y} & \frac{1}{\rho}\frac{\partial p}{\partial z} \end{bmatrix} =$$

$$\langle \frac{\partial}{\partial y}(\frac{1}{\rho}\frac{\partial p}{\partial z}) - \frac{\partial}{\partial z}(\frac{1}{\rho}\frac{\partial p}{\partial y}), \frac{\partial}{\partial z}(\frac{1}{\rho}\frac{\partial p}{\partial x}) - \frac{\partial}{\partial x}(\frac{1}{\rho}\frac{\partial p}{\partial z}), \frac{\partial}{\partial x}(\frac{1}{\rho}\frac{\partial p}{\partial y}) - \frac{\partial}{\partial y}(\frac{1}{\rho}\frac{\partial p}{\partial x})\rangle.$$

Let's concentrate on the first term of this vector:

$$\frac{\partial}{\partial y}(\frac{1}{\rho}\frac{\partial p}{\partial z}) - \frac{\partial}{\partial z}(\frac{1}{\rho}\frac{\partial p}{\partial y}) = -\frac{1}{\rho^2}\frac{\partial\rho}{\partial y}\frac{\partial p}{\partial z} + \frac{1}{\rho}\frac{\partial^2 p}{\partial y\partial z} + \frac{1}{\rho^2}\frac{\partial\rho}{\partial z}\frac{\partial p}{\partial y} - \frac{1}{\rho}\frac{\partial^2 p}{\partial z\partial y}$$

which simplifies to

$$-\frac{1}{\rho^2}(\frac{\partial\rho}{\partial y}\frac{\partial p}{\partial z} - \frac{\partial\rho}{\partial z}\frac{\partial p}{\partial y})$$

which is the first component of the vector $-\frac{1}{\rho^2}\nabla\rho \times \nabla p$. The rest of the proof follows in the same manner.

(b) Use the above identity, the conclusion in 6(a), and the Stokes theorem to compute the line integral $\oint_C \frac{1}{\rho}\nabla p \cdot d\mathbf{r}$ where C is a closed curve and p and ρ are the pressure and density of a barotropic flow.

Solution: According to the Stokes Theorem

$$\oint_C \frac{1}{\rho}\nabla p \cdot d\mathbf{r} = \int\int_S \nabla \times (\frac{1}{\rho}\nabla p)\, d\mathbf{A},$$

where S is a surface with C as its boundary. Using the result of 7(a), the latter surface integral equals

$$-\int\int_S \frac{1}{\rho^2}\nabla\rho \times \nabla p\, d\mathbf{A}.$$

But when the fluid is barotropic, the gradients of ρ and p are parallel, hence $\nabla\rho\times\nabla p = \mathbf{0}$. So

$$\oint_C \frac{1}{\rho}\nabla p \cdot d\mathbf{r} = 0.$$

8. Consider the following heat conduction problem:

$$u_t = 5u_{xx}, \qquad u(0,t) = u(2,t) = 0, \qquad u(x,0) = 1.3\sin\frac{\pi x}{2} + 3.4\sin 2\pi x.$$

(a) Describe a physical model for which the above BVP makes sense.

Solution: Heat conduction in a bar of length 2 whose ends are kept in ice and ahose initial temperature distribution is given by $1.3\sin\frac{\pi x}{2} + 3.4\sin 2\pi x$.

(b) Assuming that u is the temperature, find the units of the physical quantity whose value is 5 in the heat equation.

Solution: $\frac{u_t}{u_{xx}}$ has dimensions of

$$\frac{\mathrm{K/T}}{\mathrm{K/L^2}} = \frac{\mathrm{L^2}}{\mathrm{T}},$$

where K is the dimension of temperature, T of time, and L of distance.

(c) Use separation of variables and find the solution to this problem.

Solution: Have done separation of variables too many times in other solution sets and will not repeat it here. The answer is

$$u(x,t) = 1.3e^{-\frac{5\pi^2 t}{2}}\sin\frac{\pi x}{2} + 3.4e^{-\frac{80\pi^2 t}{2}}\sin 2\pi x.$$

(d) Use the first nonzero term of the above solution and estimate how long it takes for the temperature at $x = 1$ to reach 50 per cent of its original value.

Solution: $u(1,0) = 1.3\sin\frac{\pi}{2} + 3.4\sin 2\pi = 1.3$. So we need to find t so that

$$1.3e^{-\frac{5\pi^2 t}{2}}\sin\frac{\pi}{2} = 0.65.$$

Using logarithms, we get $t = \frac{2}{5\pi^2}\ln 2$.

9. Let $\mathbf{v} = \langle y^2, 2xy\rangle$ be the velocity field of a fluid.

(a) Compute the vorticity of the flow. Is the flow irrotational anywhere in the xy-plane?

Solution: $\nabla \times \langle y^2, 2xy\rangle = \langle 0, 0\rangle$.

(b) Compute the acceleration $\mathbf{a}$ of this flow. Does $\mathbf{a}$ have a potential p? If yes, find it.

Solution: $\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = \langle 4\,x\,y^2, 4\,x^2\,y + 2\,y^3\rangle$. To check to see if this vector has a potential we compute its curl:

$$\nabla \times \langle 4\,x\,y^2, 4\,x^2\,y + 2\,y^3\rangle = \langle 0, 0\rangle.$$

Since the curl is zero, $\mathbf{a}$ has a potential p. To find p, we set up the expressions

$$\frac{\partial\phi}{\partial x} = 4\,x\,y^2, \quad \frac{\partial\phi}{\partial y} = 4x^2y + 2y^3.$$

After integrating this expressions in the standard manner, we get

$$\phi = 2x^2y^2 + \frac{y^4}{2}.$$

10. Consider the viscous geostrophic equations listed on the previous page. Assuming that u and v are only functions of z, that ρ, f, and A_V are constants, and that $\lim_{z\to\infty} u(z) = U$, a constant, and $\lim_{z\to\infty} v(z) = 0$,

    (a) Prove that ∇p must be a constant vector.

    Solution: The equations of motion are:

$$c) \quad -fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + A_V\frac{\partial^2 u}{\partial z^2}, \quad fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + A_V\frac{\partial^2 v}{\partial z^2}, \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g.$$

    Since ρ and g are constants, the third equation implies that $\frac{\partial p}{\partial z}$ is constant. Differentiate the third equation with respect to x and y to get:

$$\frac{\partial^2 p}{\partial x \partial z} = \frac{\partial^2 p}{\partial y \partial z} = 0.$$

    Thus, $\frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial y}$ are independent of z. To see that they are independent of x and y as well (and therefore they are constant), note that the first equation states that

$$\frac{\partial p}{\partial x} = f\rho v - \rho A_V \frac{\partial^2 u}{\partial z^2}.$$

    Since u and v are functions of z only, differentiating the above equation with respect to x and y yields zero. Thus, $\frac{\partial^2 p}{\partial x^2}$ and $\frac{\partial^2 p}{\partial x \partial y} = 0$. Thus $\frac{\partial p}{\partial x}$ is independent of x and y. Since we have already shown that this function is independent of z, $\frac{\partial p}{\partial x}$ is constant. In a similar manner, we have that $\frac{\partial p}{\partial y}$ is constant. From the conditions at $z \to \infty$ we have (why?)

$$\nabla p = \langle 0, f\rho U, -\rho g\rangle.$$

    (b) Find the ODE that u must satisfy.

    Solution: Following the above computations, the geostrophic equations become

$$-fv = A_V u'', \quad fu = fU + A_V v''.$$

    Differentiate the first equation twice with respect to z, use the second equation to eliminate v and get

$$A_V^2 u'''' + f^2 u = f^2 U.$$