SM311O Project: Shallow Water Theory.
Poincaré, Kelvin, and Rossby Waves

1 Introduction

In this project, we will derive a collection of simple differential equations from the basic principles of ocean dynamics. In addition to our text, Chapter 3 of Joe Pedlosky's book (which is on reserve in the Nimitz library) is an excellent complementary source for the problems you will be asked to study in this project.

We envision a body of fluid of constant density ρ occupying the region $D = \{(x, y, z) | \, H_B(x, y) < z < H_F(x, y)\}$, where H_B defines the bottom of the ocean and H_F the air-ocean interface (free surface). We will think of the x-axis as pointing in the east direction, the y-axis pointing north, and the z-axis perpendicular to east and north in a right-handed coordinate system.

We assume that the body of fluid in D is rotating about the z-axis with constant angular velocity Ω. Let $p(x, y, z, t)$ denote the pressure in the fluid, g the constant of acceleration, p_0 the atmospheric pressure (which we assume is constant), f the coriolis parameter (typically constant, or at most changing with y, the latitude), and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ the velocity of a typical fluid particle.

2 Equations of Motion

The quations of motion are derived from conservation of mass and the balance of linear momentum. Assuming that the fluid is incompressible, conservation of mass requires that the divergence of $\mathbf{v}$ be zero or

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0. \tag{1}$$

Conservation of linear momentum, or the Navier-Stokes equations, require that

$$\rho(\frac{\partial v_1}{\partial t} - f v_2) = -\frac{\partial p}{\partial x}, \quad \rho(\frac{\partial v_2}{\partial t} + f v_1) = -\frac{\partial p}{\partial y}, \quad 0 = -\frac{\partial p}{\partial z} - \rho g. \tag{2}$$

Remark 1: We will study the derivation of equations (2) in detail later in the semester. The form of these equations, however, should not be very unfamiliar to you. These equations are the equivalent of Newton's second law of motion, that mass times acceleration must equal force. In place of mass, we are using the density of the fluid, namely, ρ. We are also using the term $\frac{\partial v_1}{\partial t}$ as an approximation for the acceleration in the eastern direction (and we will see later that we are ignoring an important **convective** term of the acceleration, but this assumption is justifiable in "shallow" waters) and $-fv_2$ as the contribution of the coriolis force to the acceleration in the horizontal direction. In the second equation in (2), the terms $\frac{\partial v_2}{\partial t}$ and fv_1 have similar interpretations. The terms $-\frac{\partial p}{\partial x}$ and $-\frac{\partial p}{\partial y}$ in the first and second equations in (2) are the contributions of the pressure gradient to the forces acting on the fluid in the eastern and northern directions. Finally, the last equation in (2), often called the **hydrostatic equation**, balances the component of pressure gradient in the z direction with the weight of the fluid. Note that we are ignoring the component of acceleration in the z-direction in comparison with the pressure gradient and the weight of the fluid.

3 Problems

Throughout these problems we will make several simplifying assumptions starting with:

(A) $\mathbf{v_1}$ **and** $\mathbf{v_2}$ **are only functions of x and y**.

(B) **Atmospheric pressure** p_0 **and density** ρ **are constant.**

3.1 Due Feb 26

We now make the additional assumptions that

$$H_B = \text{const.} \qquad \text{and} \quad v_3(x, y, 0, t) = 0. \tag{3}$$

1. Show that

$$v_3(x, y, z, t) = -z\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}\right). \tag{4}$$

(Hint: Start with (1) and use (3).)

2. A surface S is called a **free surface** if it is always occupied by the same fluid particles. The interface between the fluid and air, and the bottom of the ocean are examples of free surfaces. Let the air-fluid interface H_F be defined by a function η as follows:

$$z = H_0 + \eta(x, y, t), \tag{5}$$

where H_0 is constant. Show that for η to be a free surface it must satisfy

$$v_3\big|_{z=H_0+\eta} = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} v_1 + \frac{\partial \eta}{\partial y} v_2. \tag{6}$$

(Hint: Read page 52 of our text. Let $(x(t), y(t), z(t))$ be a typical point on the interface. Note that $z(t) = H_0 + \eta(x(t), y(t), t))$. Use this identity in t to get (6).)

3. Show that η must satisfy the relation

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left((H_0 + \eta)v_1\right) + \frac{\partial}{\partial y}\left((H_0 + \eta)v_2\right) = 0. \tag{7}$$

(Hint: Evaluate (4) at $z = H_0 + \eta$, then eliminate v_3 from the latter and (6) to get (7).)

4. We impose the following **linearization** assumption: We assume that the quadratic terms ηv_1 and ηv_2 in (7) are small compared with the linear terms $H_0 v_1$ and $H_0 v_2$. We will set them equal to zero to get

$$\frac{\partial \eta}{\partial t} + H_0 \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \right) = 0. \tag{8}$$

5. Consider the third equation in (2), the hydrostatic equation. By integrating this equation with respect to z and evaluating the result at $z = H_0 + \eta(x, y, t)$, show that the fluid pressure p is related to the atmospheric pressure as

$$p(x, y, z, t) = p_0 + \rho g(H_0 + \eta(x, y, t) - z).$$

Substitute this result into the first two equations in (2) to get the following set of equations for v_1 and v_2:

$$\frac{\partial v_1}{\partial t} - f v_2 = -g \frac{\partial \eta}{\partial x}, \qquad \frac{\partial v_2}{\partial t} + f v_1 = -g \frac{\partial \eta}{\partial y}. \tag{9}$$

You have now proved the following theorem:

Theorem 1: The triple $(v_1(x,y,t), v_2(x,y,t), \eta(x,y,t))$ satisfies the set of three partial differential equations (8) and (9).

3.2 Due March 5th

The goal of this part of the project is to derive a single equation for η from (8) and (9).

1. Define
$$\omega_1 = \frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x}, \qquad \omega_2 = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}. \tag{10}$$
Note the relationship between ω_1 and ω_2 and the vorticity and the divergence of $\mathbf{v}$, respectively. By manipulating (8) and (9), show that
$$\frac{\partial \eta}{\partial t} + H_0\omega_2 = 0, \qquad \frac{\partial \omega_2}{\partial t} + f\omega_1 = -g\Delta\eta, \qquad \frac{\partial \omega_1}{\partial t} = f\omega_2. \tag{11}$$
(Hint: The derivation of the first equation in (11) is obvious. Differentiate the first equation in (9) with respect to x, the second with respect to y and add the two resulting equations to get the second equation in (11). Next, differentiate the first equation in (9) with respect to y, the second equation with respect to x, subtract the resulting equations to get the third equation in (11).)

2. Let $u = \eta_t$. Show that u satisfies the equation
$$\frac{\partial^2 u}{\partial t^2} + f^2 u - c^2\Delta u = 0, \tag{12}$$
where $c^2 = gH_0$. (Hint: Differentiate the second equation in (11) with respect to t, use the third equation in(11) and get $\frac{\partial^2\omega_2}{\partial t^2} + f^2\omega_2 = -g\Delta u$. Now, apply the operation $\frac{\partial^2}{\partial t^2} + f^2$ to the first equation in (11) to get (12)).

The equation in (12), which is called the **wave equation**, is a fundamental equation in mathematical physics. This equation not only governs the motion of η, the free surface in this case, but the motion of many other kinds of waves, acoustic as well as electromagnetic, that one encounters in nature.