Solutions to the Homework Set on page 117

2. Look for solutions of the form $u(x,t) = F(x)T(t)$. Substitute into $u_{tt} = u_{xx}$ and divide by FT to get

$$\frac{F''}{F} = \frac{T''}{T}$$

which implies that both fractions are constant (why?). Let $-\lambda^2$ be the constant. So

$$F'' + \lambda^2 F = 0, \qquad T'' + \lambda^2 T = 0.$$

Look for solutions of the form $F(x) = e^{mx}$ for $F'' + \lambda^2 F = 0$. After substituting e^{mx} in the differential equation we get that $m^2 + \lambda^2 = 0$, or $m = \pm\lambda\mathrm{i}$. Thus a solution of our equation is $e^{\lambda i x}$ which, by Euler's formula, reduces to $\cos\lambda x + \mathrm{i}\sin\lambda x$. The real and imaginary parts of this complex solution are also solutions of the original differential equation, so

$$F(x) = c_1\cos\lambda x + c_2\sin\lambda x$$

is the general solution of $F'' + \lambda^2 F = 0$. Similarly

$$T(t) = c_3\cos\lambda t + c_4\sin\lambda t$$

is the general solution of $T'' + \lambda^2 T = 0$. Since $u(x,t) = F(x)T(t)$, the function

$$u(x,t) = (c_1\cos\lambda x + c_2\sin\lambda x)(c_3\cos\lambda t + c_4\sin\lambda t) \tag{1}$$

is the general solution of the $u_{tt} + u_{xx}$.

We now apply the boundary conditions $u(0,t) = u(1,t) = 0$ to determine the the constants in (1). First, $u(0,t) = 0$ becomes

$$c_1(c_3\cos\lambda t + c_4\sin\lambda t) = 0, \qquad \text{for all } t$$

which requires that $c_1 = 0$ (why couldn't we choose $c_3 = c_4 = 0$?). Hence, (1) reduces to

$$u(x,t) = A\sin\lambda x\cos\lambda t + B\sin\lambda x\sin\lambda t.$$

(What are A and B?) The second boundary condition, $u(1,t) = 0$, requires that

$$0 = \sin\lambda(A\cos\lambda t + B\sin\lambda t), \qquad \text{for all } t$$

which is satisfied if $\sin\lambda = 0$ or if

$$\lambda_n = n\pi.$$

Thus

$$u(x,t) = A_n\sin n\pi x\cos n\pi t + B_n\sin n\pi x\sin n\pi t$$

is the set of all normal modes of this equation. By letting $A_n = 1$, $B_n = 0$ or $A_n = 0$, $B_n = 1$, we get the two set of functions u_n and v_n stated in the answer.

3(a). $u(x,t) = 0.1\sin\pi t\sin\pi x$. See figure 1 for the snapshots of this wave. The following program generated the figure:

```
u[x_,t_] = 0.1 Sin[Pi x]Sin[Pi t];
Plot[Evaluate[Table[u[x,t], {t,0,2,0.2}]],{x,0,1}]
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3(g). $u(x,t) = \sum_{i=1} 10\frac{(-1)^i}{i} \sin i\pi t \sin i\pi x$. See figure 2 for the snapshots of this wave. The following program generated the figure:

```
u[x_,t_] = Sum[(-1)^i/i  Sin[i Pi x]Sin[i Pi t], {i, 1, 10}];
Plot[Evaluate[Table[u[x,t], {t,0,2,0.2}]],{x,0,1}]
```

4. There is little difference between this problem and Problem 2. After separating variables in $u_{tt} = 9u_{xx}$ we get

$$\frac{F''}{F} = \frac{T''}{9T} = -\lambda^2$$

which results in the two ordinary differential equations

$$F'' + \lambda^2 F = 0, \qquad T'' + 9\lambda^2 T = 0.$$

We look for solutions of the form $F(x) = e^{mx}$ and $T(T) = e^{mt}$ and find that

$$F(x) = c_1 \sin \lambda x + c_2 \cos \lambda x, \qquad T(t) = c_3 \sin 3\lambda t + c_4 \cos 3\lambda t$$

are the general solutions. So

$$u(x,t) = (c_1 \sin \lambda x + c_2 \cos \lambda x)(c_3 \sin 3\lambda t + c_4 \cos 3\lambda t)$$

is the general solution of the PDE. The first boundary condition, $u(0,t) = 0$, leads to $c_2 = 0$. So

$$u(x,t) = (A \sin 3\lambda t + B \cos 3\lambda t) \sin \lambda x.$$

The second boundary condition, $u(2,t) = 0$, gives us $\sin 2\lambda = 0$, or

$$\lambda_n = \frac{n\pi}{2},$$

for $n = 1, 2, ...$ Therefore, we end up with

$$u(x,t) = (A_n \sin \frac{3n\pi t}{2} + B_n \cos \frac{3n\pi t}{2}) \sin \frac{n\pi x}{2}.$$

Choosing $A_n = 1$, $B_n = 0$ or $A_n = 0$, $B_n = 1$, we get the necessary normal modes .

6. The only difference between this problem and Problem 2 is in the boundary conditions. Here the boundary conditions are

$$u_x(0,t) = u_x(1,t) = 0$$

which have the physical significance of allowing the string to experience displacement at its ends. After carrying out the necessary steps of the separation of variables on the PDE $u_{tt} = u_{xx}$ we end up with the general solution

$$u(x,t) = (c_1 \cos \lambda x + c_2 \sin \lambda x)(c_3 \cos \lambda t + c_4 \sin \lambda t). \tag{2}$$

To apply the first boundary condition, $u_x(0,t) = 0$, we need to first determine u_x from (2), which is

$$u_x(x,t) = (-c_1 \lambda \sin \lambda x + c_2 \lambda \cos \lambda x)(c_3 \cos \lambda t + c_4 \sin \lambda t).$$

Now $u_x(0,t) = 0$ requires that $0 = c_2\lambda(c_3 \cos \lambda t + c_4 \sin \lambda t)$ which implies that $c_2 = 0$. Hence, the general solution (2) reduces to

$$u(x,t) = (A \cos \lambda t + B \sin \lambda t) \cos \lambda x.$$

Applying the second boundary condition, $u_x(1,t) = 0$, to the above u leads to $\sin \lambda = 0$ or $\lambda_n = n\pi$, $n = 0, 1,$. **It is important to note that $n = 0$ is an eigenvalue of thsi problem**. Thus the eigenfunctions of the Neumann problem are

$$u_n(x,t) = (A_n \sin n\pi t + B_n \cos n\pi t) \cos n\pi x$$

with $n = 0, 1, 2, ...$ By letting $A_n = 1$ and $B_n = 0$, or $A_n = 0$ and $B_n = 1$, we get the normal modes $u_n = \sin n\pi t \cos n\pi x$ and $v_n = \cos n\pi t \cos n\pi x$.

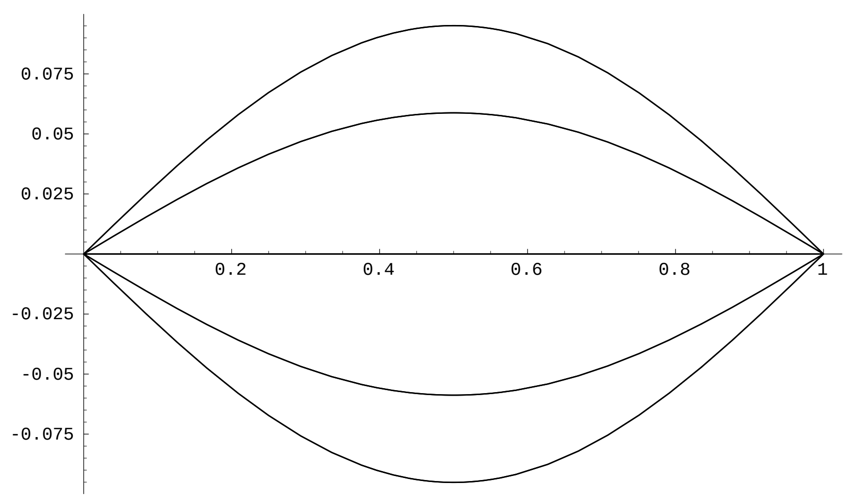

Figure 1: The snapshots of $u_1(x,t)$ (see Problem 3(a).

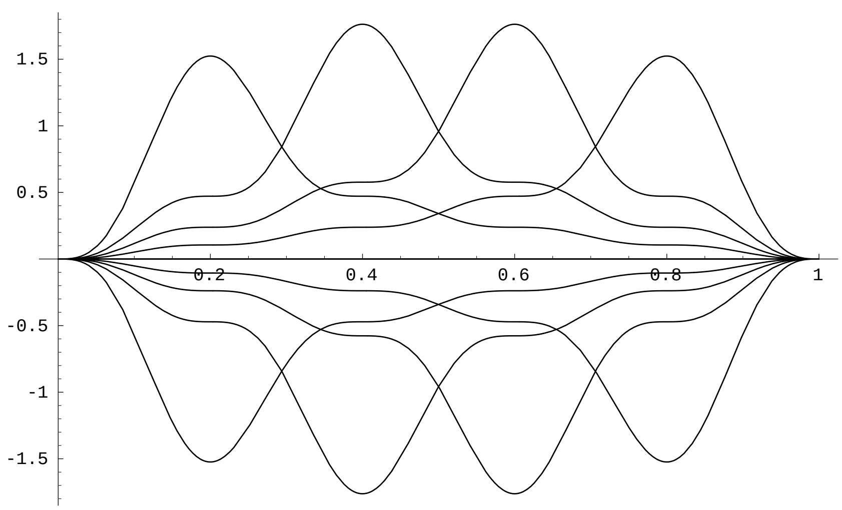

Figure 2: The snapshots of $u(x, t)$ (see Problem 3(g).