Solutions to the Homework Set on page 204

1(a): Surface S can be parametrized by

$$\mathbf{r}(u,v) = \langle u, v, -1\rangle, \qquad 0 \le u < 1, 0 \le v < 1,$$

so that

$$\mathbf{r}_u \times \mathbf{r}_v = \langle 0, 0, 1\rangle.$$

Also, since $\mathbf{v} = \langle 2x - y, y^2, 1\rangle$, then

$$\nabla \times \mathbf{v} = \langle 0, 0, 1\rangle.$$

Therefore, the flux of vorticity in this problem is

$$\int_0^1 \int_0^1 \langle 0,0,1\rangle \cdot \langle 0,0,1\rangle \, du\, dv = \int_0^1 \int_0^1 dudv = 1.$$

1(b): Note that $\mathbf{v} = \langle 2xy^2, z\rangle$ and that the vorticity of this flow, $\nabla \times \mathbf{v} = \langle 0,0,0\rangle$. Hence, the flux of vorticity is zero over any surface, including the plane that passes through $(0,0,0)$, $(1,0,0)$ and $(0,1,0)$.

1(c): To determine a parametrization for the surface S we first need to find the equation for the plane that passes through the points $(0,0,0)$, $(1,0,0)$, and $(0,1,1)$. One way to accomplish this is to note that the equation of any plane can be expressed as $Ax + By + Cz = D$ where A, B, C and D are constants. Since $(0,0,0)$ must satisfy this equation, we have $D = 0$. Similarly, $(1,0,0)$ belongs to this plane so $A = D$ which in turn requires that $A = 0$. Finally, $(0,1,1)$ belongs to this plane so $B + C = 0$, or $C = -B$. The equation of the plane now takes the form $By - Cz = 0$ or $z = y$. So the surface S can be parametrized by

$$\mathbf{r}(u,v) = \langle u, v, v\rangle,$$

where $0 \le u < 1, 0 \le v < 1 - u$ (look at the projection of S on the xy-plane). Hence,

$$\mathbf{r}_u \times \mathbf{r}_v = \langle 0, -1, 1\rangle.$$

Also, since $\mathbf{v} = \langle 2x - y, y^2, 0\rangle$, then

$$\nabla \times \mathbf{v} = \langle 0, 0, 1\rangle.$$

Therefore, the flux of vorticity in this problem is

$$\int_0^1 \int_0^{1-u} \langle 0,0,1\rangle \cdot \langle 0,-1,1\rangle \, dv\, dv = \int_0^1 \int_0^{1-u} dvdu = \frac{1}{2}.$$

2: The Stokes theorem states that

$$\int\int_S \nabla \times \mathbf{v} \cdot d\mathbf{A} = \oint_C \mathbf{v} \cdot d\mathbf{r}.$$

To verify this result with $\mathbf{v} = \langle z^2, 4x, 0\rangle$ and $S = \{(x,y,z) | -a < x < a, -a < y < a, z = c\}$ we start by noting that $\nabla \times \mathbf{v} = \langle 0, 2z, 4\rangle$ and that S has the parametrization $\mathbf{r}(u,v) = \langle u, v, c\rangle$. Note that $\mathbf{r}_u \times \mathbf{r}_v = \langle 0,0,1\rangle$. Hence, the left-hand-side of the Stokes theorem becomes

$$LHS = \int\int_S \nabla \times \mathbf{v} \cdot d\mathbf{A} = \int_{-a}^a \int_{-a}^a \langle 0, 2c, 4\rangle \cdot \langle 0,0,1\rangle \, dudv = 16a^2.$$

The line integral on the right side of Stokes theorem is the sum of four line integrals over the straight-line edges of the square. We apply the right-hand convention betwee the normal to S and for direction of traversing the curve C and parametrize these four curves as follows:

$$C_1 : \mathbf{r}(t) = \langle t, -a, c\rangle, \qquad -a < t < a.$$

$$C_2 : \mathbf{r}(t) = \langle a, t, c\rangle, \qquad -a < t < a.$$

$$C_3 : \mathbf{r}(t) = \langle -t, a, c\rangle, \qquad -a < t < a.$$

$$C_4 : \mathbf{r}(t) = \langle -a, -t, c\rangle, \qquad -a < t < a.$$

Then (note that $\mathbf{v} = \langle z^2, 4x, 0\rangle$)

$$\int_{C_1} \mathbf{v} \cdot d\mathbf{r} = \int_{-a}^{a} \langle c^2, 4t, 0\rangle \cdot \langle 1, 0, 0\rangle dt = 2ac^2.$$

Similarly,

$$\int_{C_2} \mathbf{v} \cdot d\mathbf{r} = \int_{-a}^{a} \langle c^2, 4a, 0\rangle \cdot \langle 0, 1, 0\rangle dt = 8a^2.$$

And

$$\int_{C_3} \mathbf{v} \cdot d\mathbf{r} = \int_{-a}^{a} \langle c^2, -4t, 0\rangle \cdot \langle -1, 0, 0\rangle dt = -2ac^2.$$

Finally,

$$\int_{C_4} \mathbf{v} \cdot d\mathbf{r} = \int_{-a}^{a} \langle c^2, -4a, 0\rangle \cdot \langle 0, -1, 0\rangle dt = 8a^2.$$

Hence, the sum of the four line integrals is $16a^2$, the same as LHS.

3: In this problem $\mathbf{v} = \langle 0, 0, x\cos z\rangle$ while S consists of three surfaces S_1, the lateral part of the cylinder, and S_2 and S_3 the top and the bottom of S (imagine cutting an empty soda can in half. Can you visualize S_1, S_2 and S_3?) S_1 can be parametrized as (why use cylindrical coordinates?)

$$\mathbf{r}(u, v) = \langle 2\cos u, 2\sin u, v\rangle, \qquad 0 < u < 2\pi, 0 < z < \frac{\pi}{2}.$$

Note that the vorticity of this flow is $\nabla \times \mathbf{v} = \langle 0, -\cos z, 0\rangle$ and

$$\mathbf{r}_u \times \mathbf{r}_v = \langle 2\cos u, 2\sin u, 0\rangle.$$

Hence the first part of left-hand side of the Stokes theorem becomes

$$LHS_1 = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \langle 0, -\cos v, 0\rangle \cdot \langle 2\cos u, 2\sin u, 0\rangle \, dvdu = -2\int_0^{\pi} \int_0^{\frac{\pi}{2}} \cos u \cos v \, dvdu = -4.$$

Next consider S_2, the "top" of this surface. Let's parametrize this surface as

$$\mathbf{r}(u, v) = \langle u\cos v, u\sin v, \frac{\pi}{2}\rangle.$$

On this surface the vorticity becomes $\langle 0, -\cos\frac{\pi}{2}, 0\rangle = \mathbf{0}$ so LHS_2, the contribution of this surface to the flux of vorticity, vanishes. Note that in this case we do not need to determine $\mathbf{r}_u \times \mathbf{r}_v$.

Finally, let us consider the contribution of S_3, the "bottom" of the surface. We parametrize this surface as

$$\mathbf{r}(u,v) = \langle v\cos u, v\sin u, 0\rangle, \qquad 0 < v < 2, 0 < u < \pi.$$

(compare this parametrization to the one we used for S_2. Why the change in the order we introduce u and v?) Note that

$$\mathbf{r}_u \times \mathbf{r}_v = -\langle 0,0,v\rangle.$$

So

$$LHS_3 = \int_0^{\pi}\int_0^2 \langle 0,-1,0\rangle \cdot \langle 0,0,v\rangle \, dudv = 0.$$

Therefore, the total flux of vorticity (the sum of LHS_1, LHS_2 and LHS_3) is -4:

$$LHS = -4$$

The right-hand side of the Stokes theorem in this case is the line integral of $\mathbf{v}$ over the four straight lines

$$C_1 : \mathbf{r}(t) = \langle -t, 0, 0\rangle, \qquad -2 < t < 2.$$
$$C_2 : \mathbf{r}(t) = \langle -2, 0, t\rangle, \qquad 0 < t < \frac{\pi}{2}.$$
$$C_3 : \mathbf{r}(t) = \langle t, 0, \frac{\pi}{2}\rangle, \qquad -2 < t < 2.$$
$$C_4 : \mathbf{r}(t) = \langle 2, 0, -t\rangle, \qquad -\frac{\pi}{2} < t < 0.$$

Let's first determine $\int_{C_1} \mathbf{v}|_{C_1} \cdot d\mathbf{r}$: Note that $\mathbf{v}|_{C_1} = \langle 0,0,-t\rangle$ so

$$\int_{C_1} \mathbf{v}\cdot d\mathbf{r} = \int_{-2}^{2} \langle 0, 0-t\rangle \cdot \langle -1,0,0\rangle \, dt = 0.$$

Similarly, $\int_{C_3} \mathbf{v}\cdot d\mathbf{r} = 0$. Let's now consider the line integral over C_2:

$$\int_{C_2} \mathbf{v}|_{C_2} \cdot d\mathbf{r} = \int_0^{\frac{\pi}{2}} \langle 0,0,-2\cos t\rangle \cdot \langle 0,0,1\rangle \, dt = -2.$$

Finally, we compute the line integral of $\mathbf{v}$ over C_4:

$$\int_{C_4} \mathbf{v}|_{C_4} \cdot d\mathbf{r} = \int_{-\frac{\pi}{2}}^{0} \langle 0,0,2\cos(-t)\rangle \cdot \langle 0,0,-1\rangle \, dt = -2.$$

So, the sum of the four line integrals is -4, the same value we obtained for LHS.

4: Let $\mathbf{v} = \langle f(x,y), g(x,y)\rangle$. Let

$$\mathbf{r}(u,v) = \langle h(u,v), k(u,v), 0\rangle$$

be the parametrization of D and

$$\mathbf{R}(t) = \langle a(t), b(t), 0\rangle$$

be the parametrization of its boundary. Note that $\nabla\times\mathbf{v} = \langle 0,0,g_x - f_y\rangle$, $\mathbf{r}_u\times\mathbf{r}_v = \langle 0,0,h_u k_v - h_v k_u\rangle$ and $\mathbf{R}'(t) = \langle a'(t), b'(t), 0\rangle$. The the right-hand side and left-hand side of the Stokes theorem reduce to

$$\int_{u_0}^{u_1}\int_{v_0}^{v_1} (g_x - f_y)(h_u k_v - h_v k_u)\, dudv = \int_{t_0}^{t_1} fa' + gb')\, dt.$$