**Solutions to the First Homework Set (page 7)**

**1(a)** :

(a) With $\mathbf{a}(t) = \langle -t, 2t, 1+3t \rangle$ and $\mathbf{b}(t) = \langle 0, t^2, 1-t \rangle$, we have

$$\mathbf{a} \cdot \mathbf{b} = 1 + 2\,t - 3\,t^2 + 2\,t^3.$$

To get this result in *Mathematica*, execute

```
a = {-t, 2 t, 1 + 3 t};
b = {0, t^2, 1 - t};
c = Expand[a . b]
```

(b) The operation of the dot product is commutative, so $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

(c) $\mathbf{a} \cdot \mathbf{a} = t^2 + 4t^2 + (1+3t)^2 = 1 + 6\,t + 14\,t^2$.

(d) $||\mathbf{a}||^2$ is the same as $\mathbf{a} \cdot \mathbf{a}$ (why?)

**1(b)** :

(a)

$$\mathbf{a} \times \mathbf{c} = \langle -\cos t - 3\,t\,\cos t, \sin t + 3\,t\,\sin t, -t\,\cos t - 2\,t\,\sin t \rangle$$

To get this result in *Mathematica*, execute

```
<<Calculus`VectorAnalysis`
CrossProduct[a, c]
```

**Note**: The quote symbol in `<<Calculus`VectorAnalysis`` is the backquote on your terminal (the key to the left of the key with number 1 on it)

(b) Because the operation of cross product is anti- commutative, that is $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, we have

$$\mathbf{c} \times \mathbf{a} = \langle \cos t + 3\,t\,\cos t, -\sin t - 3\,t\,\sin t, t\,\cos t + 2\,t\,\sin t \rangle$$

(c) Because $||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}||\,||\mathbf{b}|| \sin\theta$, where $\theta$ is the angle between the two vectors, $||\mathbf{a} \times \mathbf{a}|| = 0$.

(d) First we compute $\mathbf{a} \times \mathbf{b}$:

$$\mathbf{a} \times \mathbf{b} = \langle 2\,t - 3\,t^2 - 3\,t^3, t - t^2, -t^3 \rangle$$

so

$$||\mathbf{a} \times \mathbf{b}|| = \sqrt{t^2\,(5 - 14\,t - 2\,t^2 + 18\,t^3 + 10\,t^4)}.$$

**1(c)** :

(a)

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \langle -\left(1+2\,t-3\,t^2+2\,t^3\right)\,\sin t, \left(-1-2\,t+3\,t^2\right)\,\cos t - t^3\,\sin t,$$

$$-\left(-t+t^2\right)\left(2\,\cos t - \sin t\right)\rangle.$$

(b)

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \langle t^3\,\cos t, -t^3\,\sin t, -\left(-2t+3\,t^2+3\,t^3\right)\,\cos t - \left(-t+t^2\right)\,\sin t\rangle.$$

**1(h)** :

(a)

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = t\,\cos t - t^2\,\cos t + 2\,t\,\sin t - 3\,t^2\,\sin t - 3\,t^3\,\sin t.$$

(b)

$$(\mathbf{a}\times\mathbf{b}\cdot\mathbf{c} = \left(t-t^2\right)\,\cos t + \left(2\,t-3\,t^2-3\,t^3\right)\,\sin t.$$

**3(a)** The function $\mathbf{v}$ in this problem is an example of a vector field, a function that assigns vectors to points in $R^2$. This vector field could represent the velocity of a fluid flow: it assigns the velocity vector $\langle x^2-y^2, xy\rangle$ to a particle of fluid located at $(x,y)$. Figure 1 shows this vector field for a few points located along the line $x=y$. Since along this line the $x$ and $y$ components are equal, the horizontal component of $\mathbf{v}$, namely $x^2-y^2$, is zero. Hence, every vector is parallel to the $y$-axis. Also, since the vertical component of $\mathbf{v}$ is the product of $xy$, the farther one moves from the origin the larger this component becomes. Figure 1 was obtained by executing the following program in *Mathematica*:

```
<<Graphics`Arrow`

v[x_, y_] = {x^2 - y^2, x y];
Show[Graphics[Table[Arrow[{x, x}, {x, x}+v[x, x]],
      {x, 0, 1, 0.2}]]]
```

**4** :

(a) Our planet rotates once in every 24 hours, hence

$$\Omega = \frac{2\pi \text{ radians}}{24 \text{ hrs} \times 60 \text{ mins} \times 60 \text{ secs}} = 7.27221\times 10^{-5} \quad \frac{\text{radians}}{\text{secs}}$$

.

(b) Assuming that our planet is a perfect sphere of radius $r_0$, the location of every point on the surface of the planet is given by

$$\mathbf{r} = \langle x,y,z\rangle = \langle r_0\sin\phi\cos\theta, r_0\sin\phi\sin\theta, r_0\cos\phi\rangle$$

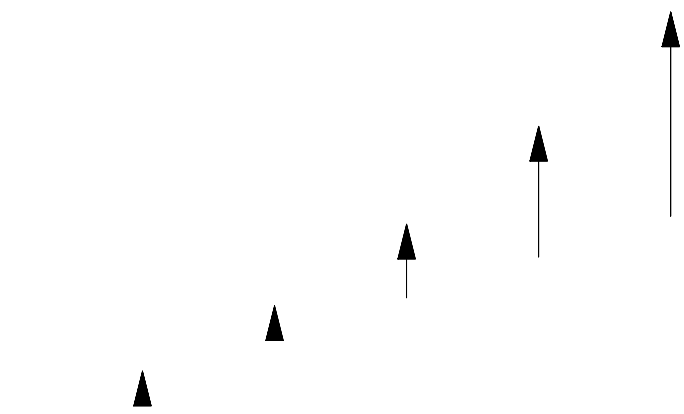

Figure 1: The vector field of Problem 3(a).

where we have used spherical coordinates. Here, $\phi$ is the latitude and $\theta$ the longitude. Similarly, the vector

$$\mathbf{\Omega} = \langle 0, 0, \Omega \rangle.$$

Then

$$\mathbf{\Omega} \times \mathbf{r} = \Omega \langle -y, x, 0 \rangle,$$

and

$$\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = -\Omega^2 \langle x, y, 0 \rangle.$$

If we define $\mathbf{R} = \langle x, y, 0 \rangle$, i.e., define $\mathbf{R}$ to be the projection of $\mathbf{r}$ on the $xy$-plane, then the centripedal acceleration is nothing but

$$-\Omega^2 \mathbf{R}.$$

Clearly, the magnitude of $\mathbf{R}$ is maximized at the equator. Assuming that our planet has a radius of $6,000$ kilometers, the value of the centripetal acceleration at the equator is

$$(7.27221 \times 10^{-5})^2 \times (6,000 \text{ kilometers}) \times (10^3 \text{ meters}) = 0.031731 \text{ m/s}^2$$

considerably smaller than the gravitational constant of $9.8 \text{ m/s}^2$. For this reason, the centripetal acceleration is often omitted from the equations of motion in ocean dynamics.