**Solutions to Homework Problems on Page 77**

1. :

   (a) $\mathbf{v} = \langle x, y \rangle$. div $\mathbf{v} = 1 + 1 = 2$. To get a representative graph of this vector field execute the following commands in *Mathematica*:

   ```
   <<Graphics`PlotField`

   v = {x, y};
   PlotVectorField[v, {x, -2, 2}, {y, -2, 2}]
   ```

   The output is shown in Figure 1.

   (b) $\mathbf{v} = \langle x, -y \rangle$. div $\mathbf{v} = 1 - 1 = 0$. See Figure 2.

   (c) $\mathbf{v} = \langle \frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}$. Then div $\mathbf{v} = \frac{\partial}{\partial x}(\frac{x}{\sqrt{x^2+y^2}}) + \frac{\partial}{\partial y}(-\frac{y}{\sqrt{x^2+y^2}})$
   $= \frac{y^2-x^2}{\sqrt{(x^2+y^2)^3}}$.
   To determine this divergence in *Mathematica* execute

   ```
   <<Calculus`VectorAnalysis`

   SetCoordinates[Cartesian[x, y, z]];
   v = {x/Sqrt[x^2+y^2], -y/Sqrt[x^2+y^2], 0};
   Simplify[Div[v]]
   ```

   See Figure 3.

   (d) $\mathbf{v} = \langle \sin y, \cos x \rangle$. Then, div $\mathbf{v} = \frac{\partial \sin y}{\partial x} + \frac{\partial \cos x}{\partial y} = 0$. See Figure 4.

   (e) $\mathbf{v} = \langle \ln\sqrt{x^2+y^2}, y \rangle$. Then, div $\mathbf{v} = 1 + \frac{x}{x^2+y^2}$. See Figure 5.

2. Let $f$ be any function of $x$, $y$, and $z$. By definition

$$\Delta f = \nabla \cdot (\nabla f).$$

   But $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$. Hence,

$$\nabla \cdot (\nabla f) = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) + \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) + \frac{\partial}{\partial z}(\frac{\partial f}{\partial z}) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

3. (a) Let $f = 3x^2 - 4y^2$. Then $\Delta f = -2$. (Try `Laplacian[3 x^2 - 4 y^2]` in *Mathematica*.

   (b) Let $f = \frac{y}{x}$. Then $\Delta f = \frac{2y}{x^2}$.

   (c) Let $f = \sin\frac{1}{x^2+y^2}$. Then $\Delta f = \frac{4}{(x^2+y^2)^3}\left((x^2+y^2)\cos\frac{1}{x^2+y^2} - \sin\frac{1}{x^2+y^2}\right)$.

   (d) Let $f = \ln(x^2+y^2)$. Then, $\Delta f = 0$.

   (e) Let $f = \frac{1}{\sqrt{x^2+y^2}}$. Then, $\Delta f = \frac{4}{(x^2+y^2)^2}$.

(f) Let $f = \arctan \frac{y}{x}$. Then, $\Delta f = 0$. (Try `Simplify[Laplacian[ArcTan[y/x]]]` in *Mathematica*.

5. (a) div $(\rho \mathbf{v}) = \rho \mathrm{div} \mathbf{v} + \mathbf{v} \cdot \nabla \rho$.

**Proof**: L.H.S. = div $(\rho bfv)$ = div $(\langle \rho v_1, \rho v_2, \rho v_3 \rangle) = \frac{\partial(\rho v_1)}{\partial x} + \frac{\partial(\rho v_2)}{\partial y} + \frac{\partial(\rho v_3)}{\partial z}$. We now use the product rule of differentiation to get

$$\text{L.H.S.} = (\frac{\partial \rho}{\partial x} v_1 + \rho \frac{\partial v_1}{\partial x}) + (\frac{\partial \rho}{\partial x} v_2 + \rho \frac{\partial v_1}{\partial y}) + (\frac{\partial \rho}{\partial x} v_3 + \rho \frac{\partial v_3}{\partial z}),$$

which can be regrouped as

$$\rho \left( \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial x} + \frac{\partial v_3}{\partial z} \right) + \left( \frac{\partial \rho}{\partial x} v_1 + \frac{\partial \rho}{\partial y} v_2 + \frac{\partial \rho}{\partial z} v_3 \right).$$

The above expression is equivalent to the R.H.S.

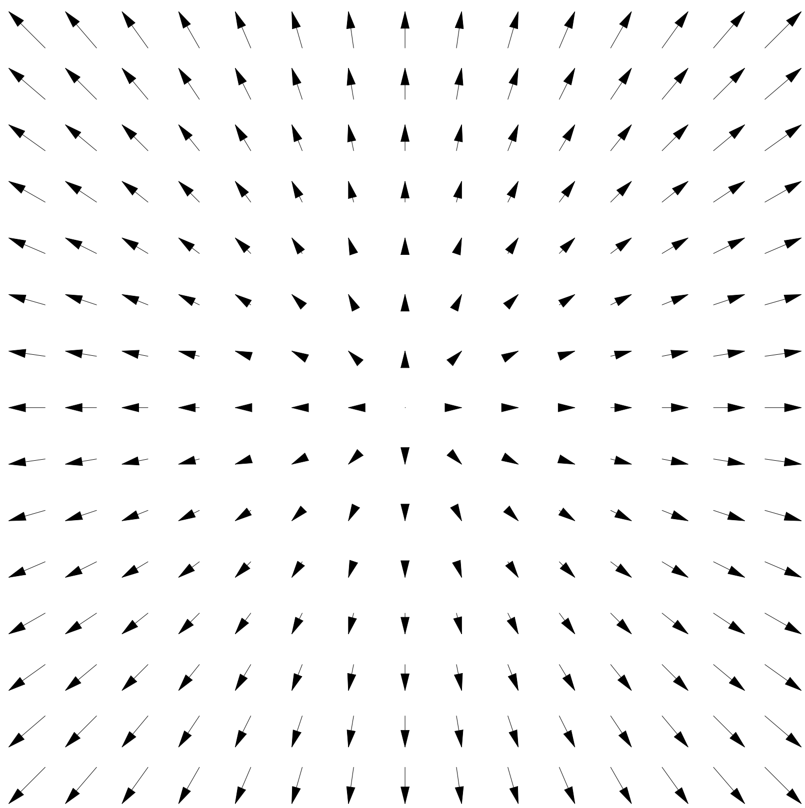

Figure 1: Vector field $\mathbf{v} = \langle x, y \rangle$.

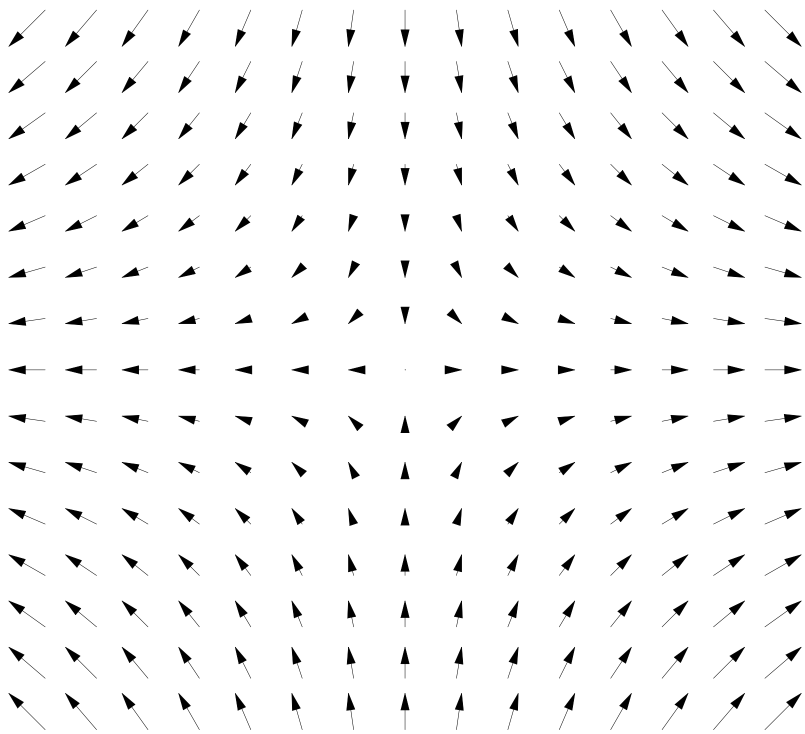

Figure 2: Vector field $\mathbf{v} = \langle x, -y \rangle$.

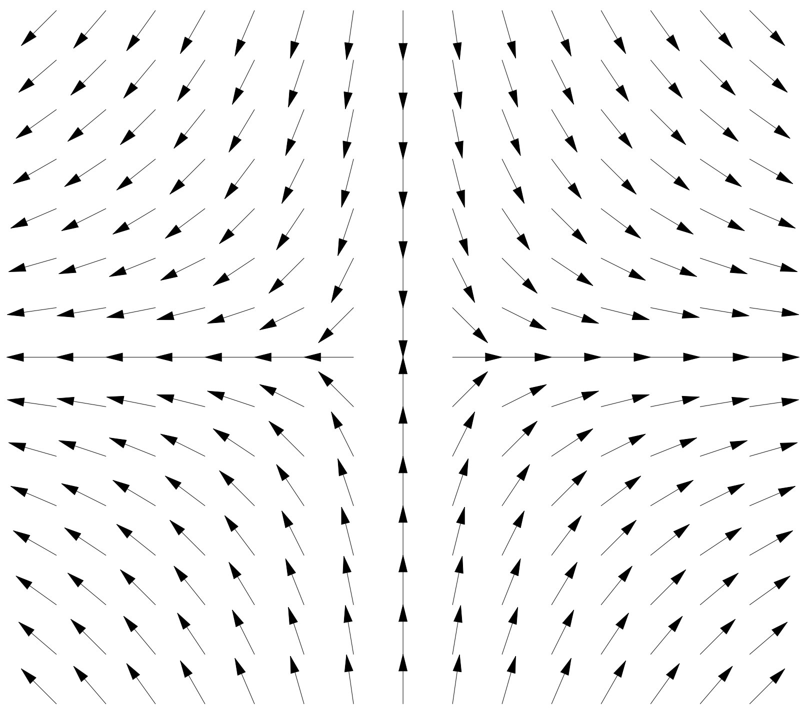

Figure 3: Vector field $\mathbf{v} = \langle \frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}} \rangle$.

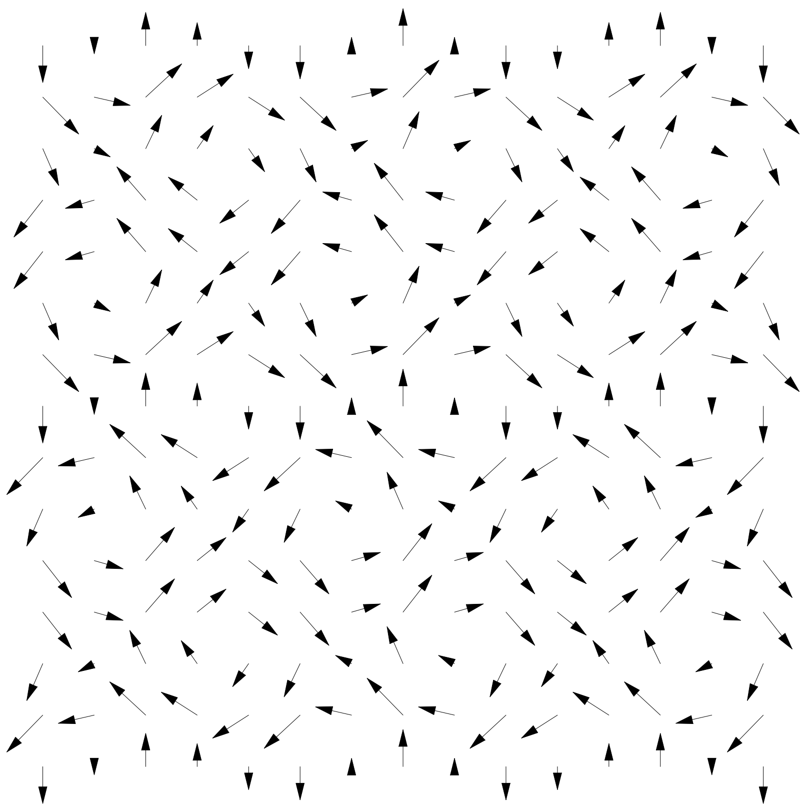

Figure 4: Vector field $\mathbf{v} = \langle \sin y, \cos x \rangle$.

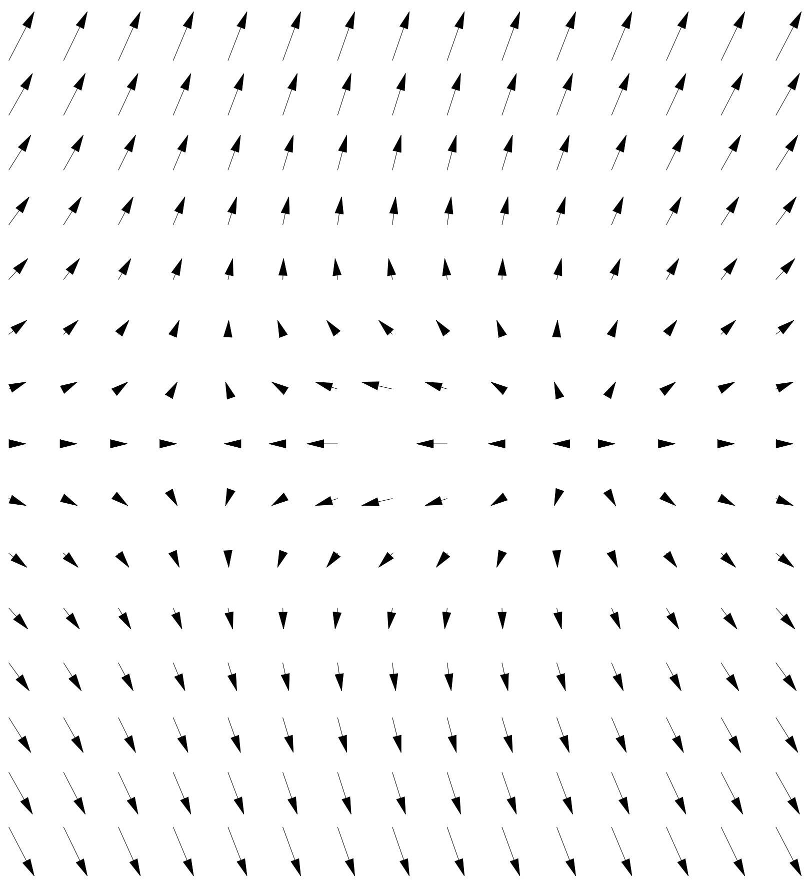

Figure 5: Vector field $\mathbf{v} = \langle \ln\sqrt{x^2+y^2}, y\rangle$.