Solutions to Homework Problems on Page 86

1. (a) $\mathbf{v} = \langle x, y\rangle$. curl $\mathbf{v} = \langle 0, 0, 0\rangle$.

   (b) $\mathbf{v} = \langle x, -y\rangle$. curl $\mathbf{v} = \langle 0, 0, 0\rangle$.

   (c) $\mathbf{v} = \langle \frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}$. Then curl $\mathbf{v} = \langle 0, 0, \frac{2xy}{\sqrt{(x^2+y^2)^3}}\rangle$.
   To determine this curl in *Mathematica* execute

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<<Calculus`VectorAnalysis`

SetCoordinates[Cartesian[x, y, z]];
v = {x/Sqrt[x^2+y^2], -y/Sqrt[x^2+y^2], 0};
Simplify[Curl[v]]
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   (d) $\mathbf{v} = \langle \sin y, \cos x\rangle$. Then, curl $\mathbf{v} = \langle 0, 0, -\cos y - \sin x\rangle$.

   (e) $\mathbf{v} = \langle \ln\sqrt{x^2+y^2}, y\rangle$. Then, curl $\mathbf{v} = \langle 0, 0, -\frac{y}{x^2+y^2}\rangle$.

2. (a) $\mathbf{v} = \langle y, -x\rangle$. Then div $\mathbf{v} = 0$ so the flow is incompressible. The vorticity of the flow is curl $\mathbf{v} = -2\mathbf{k}$. Since the flow is incompressible, a stream function ψ exists. It satisfies

$$\frac{\partial\psi}{\partial y} = y, \qquad \frac{\partial\psi}{\partial x} = x.$$

   After integrating the above equations, we get $\psi = \frac{x^2}{2} + \frac{y^2}{2}$. Its contours are shown in Figure 1.

   (b) $\mathbf{v} = \langle x, y\rangle$. Then div $\mathbf{v} = 2$ so the flow is compressible. The vorticity of the flow is curl $\mathbf{v} = \langle 0, 0, 0\rangle$. Since the flow is compressible, it does not possess a stream function.

   (c) $\mathbf{v} = \langle y, -\sin x\rangle$. Then div $\mathbf{v} = 0$ so the flow is incompressible. The vorticity of the flow is curl $\mathbf{v} = -(1+\cos x)\mathbf{k}$. Since the flow is incompressible, a stream function ψ exists. It satisfies

$$\frac{\partial\psi}{\partial y} = y, \qquad \frac{\partial\psi}{\partial x} = \sin x.$$

   After integrating the above equations, we get $\psi = -\cos x + \frac{y^2}{2}$. Its contours are shown in Figure 2.

   (d) $\mathbf{v} = \langle y^3 - x^2, -y\cos x\rangle$. Then div $\mathbf{v} = -(2x + \cos x)$ so the flow is compressible. The vorticity of the flow is curl $\mathbf{v} = -3y^2 + y\sin x)\mathbf{k}$. Since the flow is compressible, it does not possess a stream function.

   (e) $\mathbf{v} = \langle x^2 - y^2, xy\rangle$. Then div $\mathbf{v} = 3x$ so the flow is compressible. The vorticity of the flow is curl $\mathbf{v} = 3y\mathbf{k}$. Since the flow is compressible, it does not possess a stream function.

3. (a) div (curl $\mathbf{v}$) = 0.

Proof: L.H.S. = div (curl bfv) = div $(\langle \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \rangle)$, which in turn is equal to

$$\frac{\partial}{\partial x}(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}) + \frac{\partial}{\partial y}(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}) + \frac{\partial}{\partial z}(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y})$$

which is equivalent to

$$\frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_2}{\partial \partial z}) + \frac{\partial^2 v_1}{\partial y \partial z} - \frac{\partial^2 v_3}{\partial y \partial x}) + \frac{\partial^2 v_2}{\partial z \partial x} - \frac{\partial^2 v_1}{\partial z \partial y}$$

which equals zero since for each term there is an equivalent term with a negative sign. This completes the proof.

(b) $\nabla \times \nabla \phi = \mathbf{0}$.

Proof: Let ϕ be an arbitrary function of x, y, and z. Then

$$\text{L}.H.S. = \nabla \times \langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \rangle$$

which is equal to

$$\langle -\frac{\partial}{\partial z}(\frac{\partial \phi}{\partial y}), \ldots, \ldots \rangle$$

where the dots denote similar terms. But the first component of the above integral is zero because cross differentiation is a commutative operation, i.e., it does not make any difference if one differentiates a function ϕ first with respect to y and then with respect to z or vice versa. The two remaining terms in the vector, denoted by dots, also end up being zero for similar reasons. This completes the proof.

4. curl $\phi \mathbf{v}$ = grad $\phi \times \mathbf{v} + \phi$curl $\mathbf{v}$.

Proof:

$$\text{L.H.S.} = \nabla \times \langle \phi v_1, \phi v_2, \phi v_3 \rangle = \langle \frac{\partial}{\partial y}(\phi v_3) - \frac{\partial}{\partial z}(\phi v_2), \frac{\partial}{\partial z}(\phi v_1) - \frac{\partial}{\partial x}(\phi v_2), \frac{\partial}{\partial x}(\phi v_2) - \frac{\partial}{\partial y}(\phi v_1) \rangle,$$

which after applying the product rule of differentiation, leads to

$$\rangle \frac{\partial \phi}{\partial y} v_3 + \phi \frac{\partial v_3}{\partial y} - (\frac{\partial \phi}{\partial z} v_2 + \phi \frac{\partial v_2}{\partial z}), \frac{\partial \phi}{\partial z} v_1 + \phi \frac{\partial v_1}{\partial z} - (\frac{\partial \phi}{\partial x} v_2 + \phi \frac{\partial v_2}{\partial x}), \frac{\partial \phi}{\partial x} v_2 + \phi \frac{\partial v_2}{\partial x} - (\frac{\partial \phi}{\partial y} v_1 + \phi \frac{\partial v_1}{\partial y}) \rangle.$$

Next we collect all terms with ϕ in them as one term and see that the resuting coefficient is $\nabla \mathbf{v}$. Similarly all the terms with components of $\mathbf{v}$ in them can be written as $\nabla \phi \times \mathbf{v}$. This completes the proof.

5.

6. This identity is the same as the one in (c).

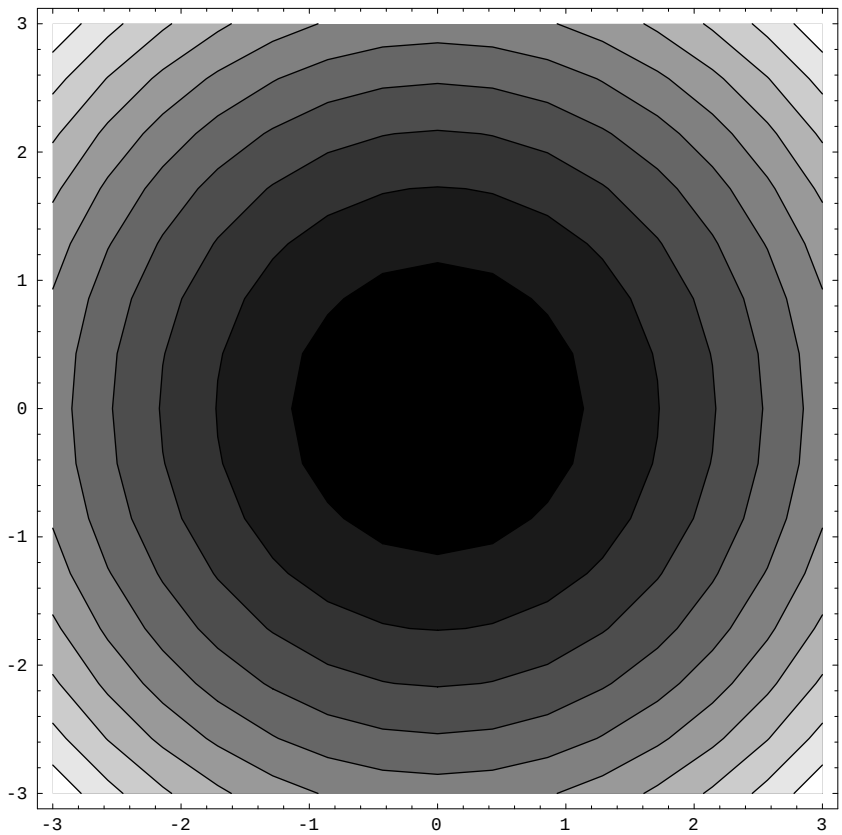

Figure 1: Contours of $\psi = \frac{x^2}{2} + fracy^2 2$.

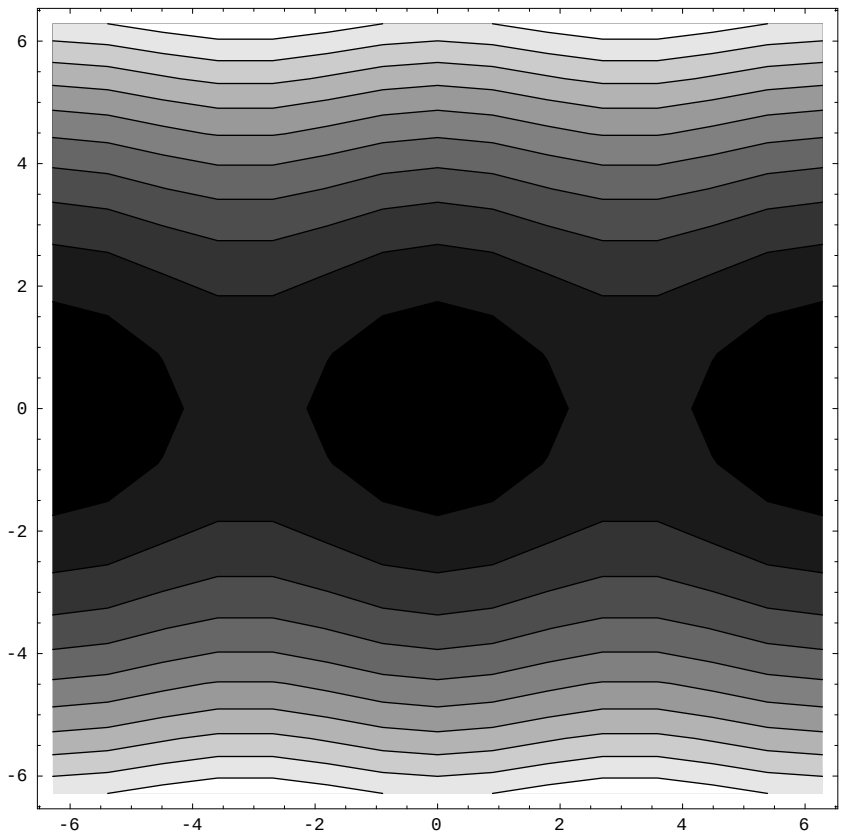

Figure 2: Contours of $\psi = -\cos x + \frac{y^2}{2}$.