

# The Navier–Stokes Equations in a Rotating Frame

## 1 Vector Representation in a Rotating Frame

One of the most important features that distinguishes most flows in fluid dynamics from those in ocean dynamics is the rotation of the Earth. In many oceanic flows, such as the Gulf Stream or typical hurricanes, the time and spatial scales are of the kind that rotation of the planet has a significant impact on the motions under study.

A second feature, and not unrelated to the first, is that equations of motion, those related to the balance of linear momentum, must be written in an inertial frame. On the other hand, nearly all of the measurements that we routinely make of observable quantities such as velocity, vorticity, salinity, pressure, among others, are made in-situ and, thus, in a rotating frame. In these notes we will describe the extra terms that need to be added to the equations of motion to account for relative motion of particles of fluid.

We start by describing the velocity of a particle that is stationary with respect to a rotating frame and is being observed from an inertial frame. Consider a rotating frame with angular velocity  $\Omega$  as shown in Figure 1. From the point of view of a stationary observer, each stationary particle  $P$  traces a path which is a circle of radius  $\bar{R}$  centered at  $(0, 0, \bar{z})$ . Assuming that  $P$ 's latitude is  $\phi$  (latitude will be measured with respect to the  $z = 0$  plane), then

$$\bar{R} = R \cos \phi, \quad \text{and} \quad \bar{z} = R \sin \phi. \quad (1)$$

Let  $\mathbf{r}(t)$  be the position vector at a typical point on the circle  $C$ . Then

$$\mathbf{r}(t) = \langle r(t) \cos \theta(t), r(t) \sin \theta(t), \bar{z} \rangle$$

where  $r$  and  $\theta$  are polar coordinates in the plane  $z = \bar{z}_0$ . Since the frame's angular velocity is  $\Omega$ , then  $\theta(t) = \Omega t$ . Returning to (1), we have

$$\mathbf{r}(t) = \langle R \cos \phi \cos \Omega t, R \cos \phi \sin \Omega t, R \sin \phi \rangle, \quad \text{with} \quad t \in \left(0, \frac{2\pi}{\Omega}\right). \quad (2)$$

Differentiating the expression in (2) with respect to  $t$  yields the tangent vector to  $C$ :

$$\mathbf{r}'(t) = \Omega \langle -R \cos \phi \sin \Omega t, R \cos \phi \cos \Omega t, 0 \rangle. \quad (3)$$

Equivalently this statement can be written as

$$\frac{d\mathbf{r}}{dt} = \boldsymbol{\Omega} \times \mathbf{r}, \quad (4)$$

where  $\boldsymbol{\Omega} = \Omega \mathbf{k}$ .

**Problem 1:**

1. Verify the statement in (4).
2. Use the result of the previous problem to show that

$$\frac{d^2\mathbf{r}}{dt^2} = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}), \quad (5)$$

3. Use the result of the previous problem to show that

$$\frac{d^2\mathbf{r}}{dt^2} = -\Omega^2 \mathbf{R},$$

where  $\mathbf{R}$  is the projection of  $\mathbf{r}$  on the  $xy$ -plane.

Note that the latter expression states that  $\mathbf{r}''$  is always pointed toward the  $z$ -axis with its magnitude largest at the equator and zero at the poles. The vector  $-\mathbf{r}''$  is called the **centrifugal** force.

Equation (4) describes the rate of change of any vector  $\mathbf{r}$  in a rotating frame when observed in an **inertial frame**. Equation (5) describes the **centripetal acceleration** experienced by a particle of fluid which remains stationary relative to the sphere.

**Problem 2:** Estimate the value of  $\mathbf{r}''$  by using  $R = 6,000$  kilometers for the radius of the Earth and the standard value for  $\Omega$ .

As shown in Problem 2, the term  $\mathbf{r}''$  is approximately  $0.03 \text{ m/sec}^2$  and is often small when compared with other terms that appear in the equation of balance of linear momentum and in typical modeling of large scale ocean dynamics, when the time scales are small (on the order of days or weeks), this term is ignored.

## 2 Velocity and Acceleration in a Rotating Frame

A typical fluid particle not only moves relative to a fixed observer in an inertial frame, it also moves relative to the particles near it. If  $\mathbf{r}(t)$  represents the trajectory of a particle, then its velocity  $\mathbf{r}'$ , as measured by an observer in an inertial frame, is the sum of  $\mathbf{v}$ , the fluid particle's velocity relative to the planet (or an observer that is stationary on the planet), and  $\boldsymbol{\Omega} \times \mathbf{r}$ , the change in the representation of the particle's position  $\mathbf{r}$  due to the rotation of the Earth. Thus, the **absolute** velocity of a fluid particle is

$$\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}. \quad (6)$$

The vector  $\mathbf{v}$  is typically what we measure in the oceans so our concern here is to write down the equations of motion of ocean dynamics in terms of this vector.

Let  $D_i$  denote the time-rate of change of a vector in an inertial frame. Let  $D_r$  denote the time-rate of change of the same quantity in a rotating frame. Then, following the discussion that led to (6), we have

$$D_i = D_r + \boldsymbol{\Omega} \times \quad . \quad (7)$$

In this notation, equation (6) becomes

$$D_i \mathbf{r} = D_r \mathbf{r} + \boldsymbol{\Omega} \times \mathbf{r}. \quad (8)$$

We now apply  $D_i$  to  $D_i \mathbf{r}$  to obtain the **absolute** acceleration of a particle in terms of its **relative** acceleration. Following (8) we have

$$\text{Absolute acceleration} = D_i(D_i \mathbf{r}) = D_r(D_i \mathbf{r}) + \boldsymbol{\Omega} \times D_i \mathbf{r}. \quad (9)$$

Using (8) again, we have

$$\text{Absolute acceleration} = D_r(D_r \mathbf{r} + \boldsymbol{\Omega} \times \mathbf{r}) + \boldsymbol{\Omega} \times (D_r \mathbf{r} + \boldsymbol{\Omega} \times \mathbf{r}). \quad (10)$$

Assuming that  $\boldsymbol{\Omega}$  does not change with time, the above equation reduces to

$$\text{Absolute acceleration} = D_r^2 \mathbf{r} + 2\boldsymbol{\Omega} \times D_r \mathbf{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}). \quad (11)$$

The first term on the right-hand side of (11) is  $\mathbf{a}$ , the acceleration of the fluid particle measured in the rotating frame. The term  $D_r \mathbf{r}$  in the second term is  $\mathbf{v}$ , the velocity of the particle in the rotating frame. With this in mind, the second term becomes

$$2\boldsymbol{\Omega} \times \mathbf{v} \quad (12)$$

and is called the **Coriolis force**. The third term in (11) is the centripetal acceleration that we encountered earlier.

The Coriolis force is an **apparent** force that appears in the description of the acceleration because of the rotation of the Earth. It is not a force in the traditional physical sense. Perhaps the best way to interpret this “force” is to imagine standing at the North Pole and observing a projectile’s path which is shot in the direction of a specific meridian. Because of the rotation of the Earth, the projectile will appear to be veering to the right of the observer. The term  $2\boldsymbol{\Omega} \times \mathbf{v}$  is the “force” that acts in the perpendicular direction to  $\mathbf{v}$  and is responsible in moving the projectile to the right.

### 3 Absolute and Relative Vorticity

As discussed above, the absolute velocity of a fluid particle is the sum of its relative velocity  $\mathbf{v}$  and  $\boldsymbol{\Omega} \times \mathbf{r}$ . Therefore, the absolute vorticity of a fluid particle is

$$\nabla \times (\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}) = \omega + 2\boldsymbol{\Omega}, \quad (13)$$

where  $\omega$  is the relative vorticity.

**Problem 3:** Verify the statement in (13).

The vector  $2\boldsymbol{\Omega}$  is the vorticity imparted to a fluid particle due to the rotation of the earth. In many large scale motions in the oceans,  $2\boldsymbol{\Omega}$  is the substantial part of the total vorticity experienced by a fluid particle.

### 4 The Coriolis Force in Component Form

Let  $P$  be a fixed but arbitrary point having  $\phi$  for its latitude. Let  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  be a set of unit vectors at  $P$  with  $\mathbf{e}_1$  pointing in the east direction,  $\mathbf{e}_2$  pointing north, and  $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$ . Let

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3. \quad (14)$$

**Problem 4:** Show that  $\boldsymbol{\Omega}$  is in the same plane that contains  $\mathbf{e}_2$  and  $\mathbf{e}_3$  and that, in fact,

$$\boldsymbol{\Omega} = \Omega(\cos \phi \mathbf{e}_2 + \sin \phi \mathbf{e}_3). \quad (15)$$

Since  $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$ , with similar relations holding among other combinations of these vectors, we have

$$\begin{aligned} \boldsymbol{\Omega} \times \mathbf{v} &= \Omega(\cos \phi \mathbf{e}_2 + \sin \phi \mathbf{e}_3) \times (v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) = \\ &= (v_3 \cos \phi - v_2 \sin \phi)\mathbf{e}_1 + v_1 \sin \phi \mathbf{e}_2 - v_1 \cos \phi \mathbf{e}_3. \end{aligned} \quad (16)$$

The quantity

$$f = 2\Omega \sin \phi$$

is called the **Coriolis paramter**. With the **assumption** that  $v_3$  is generally small on the average when compared with  $v_1$  and  $v_2$ , we eliminate terms involving  $v_3$  from (16). Also, we eliminate the  $\mathbf{e}_3$  term in (16) in comparison with the other terms that appear in balance of linear momentum in the  $\mathbf{e}_3$  direction. We end up with

$$2\boldsymbol{\Omega} \times \mathbf{v} = -fv_2\mathbf{e}_1 + fv_1\mathbf{e}_2. \quad (17)$$

## 5 Equations of motion

The equations of balance of linear momentum are

$$\begin{aligned} \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_1}{\partial y} + v_3 \frac{\partial v_1}{\partial z} - v_2 f &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + F_1 \\ \frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial v_2}{\partial y} + v_3 \frac{\partial v_2}{\partial z} + v_1 f &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + F_2 \\ \frac{\partial v_3}{\partial t} + v_1 \frac{\partial v_3}{\partial x} + v_2 \frac{\partial v_3}{\partial y} + v_3 \frac{\partial v_3}{\partial z} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} - g + F_3 \end{aligned} \quad (18)$$

These equations are augmented by the the equation of conservation of mass, which in the case of an incompressible fluid states

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0. \quad (19)$$

Vector  $\mathbf{F}$  represents the viscous forces in the medium.

We saw in our discussion of viscous forces that

$$\mu \frac{\partial^2 u}{\partial z^2}$$

is often used to model viscous forces in a fluid flow. Based on scale arguments, in many oceanic flows the following representations are used

$$F_1 = A_V \frac{\partial^2 v_1}{\partial z^2}, \quad F_2 = A_V \frac{\partial^2 v_2}{\partial z^2}, \quad F_3 = 0.$$

When the flow is steady and the acceleration terms are small compared with the Coriolis force, the pressure gradient, and viscous forces, we end up with the viscous **Geostrophic** equations

$$\begin{aligned} -v_2 f &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + A_V \frac{\partial^2 v_1}{\partial z^2} \\ v_1 f &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + A_V \frac{\partial^2 v_2}{\partial z^2} \\ 0 &= -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \end{aligned} \quad (20)$$

## 6 Ekman Layer

Problem: Find a solution to (20) subject to the following conditions:

1.  $v_1 = u(z)$  and  $v_2 = v(z)$ .
2.  $u = U$  and  $v = 0$  when  $z \rightarrow \infty$ .
3.  $u = v = 0$  when  $z = 0$ .
4.  $\rho$  and  $A_V$  are constants.

Equations (20) now take the form

$$\begin{aligned} -fv &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + A_V \frac{\partial^2 u}{\partial z^2} \\ fu &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + A_V \frac{\partial^2 v}{\partial z^2} \\ g &= -\frac{1}{\rho} \frac{\partial p}{\partial z} \end{aligned} \quad (21)$$

Note that if we differentiate (21.3) with respect to  $x$  and  $y$  we get that  $\frac{\partial^2 p}{\partial z \partial x}$  and  $\frac{\partial^2 p}{\partial z \partial y}$  are zero which imply that the horizontal pressure gradient  $\langle \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y} \rangle$  is independent of  $z$ . From the first two equations in (21) it follows that these quantities are independent of  $x$  and  $y$  as well since  $u$  and  $v$  only depend on  $z$ . Thus  $\langle \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y} \rangle$  is a constant vector. On the other hand, we expect that as  $z \rightarrow \infty$  the quantities  $u''(z)$  and  $v''(z)$  approach zero because  $u$  and  $v$  have asymptotic values 0 and  $U$ , respectively (it might be helpful to draw graphs of typical  $u$  and  $v$  in order to see this point). It then follows from the first two equations in (21) that

$$\frac{1}{\rho} \frac{\partial p}{\partial x} = 0, \quad \frac{1}{\rho} \frac{\partial p}{\partial y} = -\rho f U. \quad (22)$$

Next, define  $\tilde{u} = u - U$ . Then the equations of motion become

$$-fv = A_V \frac{\partial^2 \tilde{u}}{\partial z^2}, \quad f\tilde{u} = A_V \frac{\partial^2 v}{\partial z^2}. \quad (23)$$

Eliminating  $v$  from these equations, we have

$$\tilde{u}'''' + \alpha^2 \tilde{u} = 0, \quad \text{where } \alpha^2 = \frac{f^2}{A_V^2}. \quad (24)$$

We seek solution to (24) of the form  $u(z) = e^{mz}$ .

**Problem 5:**

1. Show that  $m$  is the root of the 4-th order auxiliary polynomial

$$m^4 + \alpha^2. \quad (25)$$

2. Show that each root of (25) is equivalent to  $m^2 = \pm\alpha i$  and that

$$m = \pm\sqrt{\frac{\alpha}{2}}(1 + i), \quad m = \pm\sqrt{\frac{\alpha}{2}}(-1 + i). \quad (26)$$

(Start by checking directly that  $i = (\frac{1}{\sqrt{2}}(1 + i))^2$ .)

Let

$$\gamma = \sqrt{\frac{\alpha}{2}}.$$

**Problem 6:** Show that the four roots of  $m$  give rise to the following general solution of (24):

$$u(z) = U + c_1 e^{-\gamma z} \cos \gamma z + c_2 e^{-\gamma z} \sin \gamma z + c_3 e^{\gamma z} \cos \gamma z + c_4 e^{-\gamma z} \sin \gamma z. \quad (27)$$

Because we are only interested in solutions that are bounded as  $z \rightarrow \infty$  we choose  $c_3 = c_4 = 0$ . Hence,

$$u(z) = U + c_1 e^{-\gamma z} \cos \gamma z + c_2 e^{-\gamma z} \sin \gamma z. \quad (28)$$

Now the boundary condition at  $z = 0$  requires that  $u(0) = 0$ . But from (28)  $u(0) = U + c_1$ . Hence

$$u(z) = U + -U e^{-\gamma z} \cos \gamma z + c_2 e^{-\gamma z} \sin \gamma z. \quad (29)$$

Also, from (23) and the boundary condition  $v(0) = 0$ , we have that  $u''(0) = 0$ . Differentiating (29) twice with respect to  $z$  and setting  $z = 0$  yields  $c_2 = 0$ . Thus

$$u = U(1 - e^{-\gamma z} \cos \gamma z),$$

from which and (23) we have (recalling the relations between  $\alpha$ ,  $f$ , and  $A_V$ )

$$v = U e^{-\gamma z} \sin \gamma z.$$

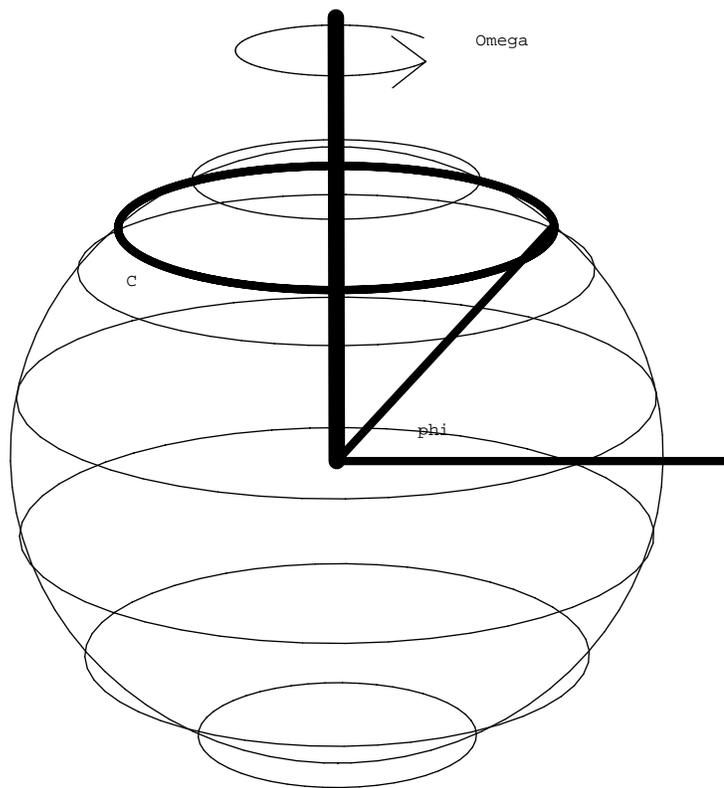


Figure 1: A rotating frame.