Project One

Velocity Fields in Rectangular and Polar Coordinates

Many fluid flows and currents in oceanography (or geophysical fluid dynamics) have a predominantly circular character to them. Hurricanes, tornadoes and streams, such as the Gulf Stream, are examples of flows whose structure is dominated by a vortex or a gyre. When the flow is two-dimensional, it is often mathematically more economical to represent these flows in polar coordinates rather than in rectangular coordinates. The purpose of this project is to develop the mathematical representations of velocity in these two coordinate systems.

Let $\mathbf{v}(x, y)$ be a velocity field. We represent this velocity in rectangular coordinates as

$$\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the horizontal and vertical directions. With $(x(t), y(t))$ denoting the path of a fluid particle, we have the system of coupled differential equations that relate the position of a particle at time t to its velocity:

$$\frac{dx}{dt} = v_1(x, y, t), \quad \frac{dy}{dt} = v_2(x, y, t). \tag{1}$$

When the original position of the particle is known (such as $x(0) = 1, y(0) = -2$, say), one solves the system of differential equations in (1) to get the particle path.

We represent the velocity of a particle in polar coordinates by

$$\mathbf{v} = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta \tag{2}$$

where $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are unit vectors in the directions of increasing r and θ, see Figure 1. The goal of this project is to find the analogue of (1) in terms of $r(t)$ and $\theta(t)$, the polar coordinates of $(x(t), y(t))$.

1. Recall the polar–rectangular relations

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{3}$$

Use these relations to show that

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \qquad \mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}. \tag{4}$$

Use these relations to show

$$\mathbf{i} = \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta, \qquad \mathbf{j} = \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta. \tag{5}$$

Moreover, show that

$$\frac{d\mathbf{e}_r}{d\theta} = \mathbf{e}_\theta, \qquad \frac{d\mathbf{e}_\theta}{d\theta} = -\mathbf{e}_r.$$

Remark 1: Note that the subscripts r and θ in e_r and e_θ do not denote partial differentiations!

Remark 2: It is important to note that the unit vectors i and j do not depend on x and y, the position at which they apply. Hence, $\frac{\partial i}{\partial x} = \frac{\partial i}{\partial y} = 0$, with similar relations holding for j. On the other hand, the unit vectors e_r and e_θ depend on θ of the position at which they apply.

2. Let $(x(t), y(t))$ and $(r(t), \theta(t))$ denote the parametrization of the path of a particle, the former in rectangular coordinates and the latter in polar coordinates. It follows form (3) that

$$x(t) = r(t)\cos\theta(t); \qquad y(t) = r(t)\sin\theta(t). \tag{6}$$

Returning to (1), we have the following relation between v and x and y:

$$v = \frac{dx}{dt} i + \frac{dy}{dt} j.$$

Use this relation and (6) to show that

$$v = \frac{dr}{dt} e_r + r\frac{d\theta}{dt} e_\theta. \tag{7}$$

Compare (7) and (2) to conclude that

$$\frac{dr}{dt} = v_r; \qquad \frac{d\theta}{dt} = \frac{1}{r} v_\theta. \tag{8}$$

Systems (1) and (8) are two representations of the same fluid flow. Because v_r and v_θ are functions of r and theta, system (8) defines a system of differential equations in $r(t)$ and $\theta(t)$ that, when combined with (6), leads to the same path $(x(t), y(t))$ that one would get from (1).

3. (a) Consider the Merry-go-round velocity field $v = yi - xj$. Show that $v = -re_\theta$. (Hint: Use (5).)

   (b) Let $P = (-1, 0)$ be the position occupied by a fluid particle at time $t = 0$. Find the path of this particle under v twice, once by solving (1) and next by solving (8). Plot the two particle paths. Color the path in rectangular coordinates red and the one in polar coordinates blue. Combine the two graphs to see that they are identical.

4. (a) Consider the Oseen vortex $v = \frac{y}{\sqrt{x^2+y^2}} i - \frac{x}{\sqrt{x^2+y^2}} j$. Show that $v = -e_\theta$.

   (b) Plot the path of the particle located at $(1, 0)$ at time 0 for $t \in (0, \infty)$, first using the rectangular representation and next the polar representation.

5. (a) Consider the Line vortex $v = \frac{y}{x^2+y^2} i - \frac{x}{x^2+y^2} j$. Show that $v = -\frac{1}{r} e_\theta$.

   (b) Plot the path of the particle located at $(1, 0)$ at time 0 for $t \in (0, \infty)$, first using the rectangular representation and next the polar representation.

6. (a) Using the Merry-go-round velocity field, plot the paths of the particles located at $(-1, 0)$, $(-2, 0)$, $(-3, 0)$, and $(-4, 0)$ at time zero for $t \in (0, \infty)$. Color the particle paths red and combine them on the same screen.

   (b) Repeat the above problem for the Oseen and Line vortices, coloring the particle paths green and blue, respectively.

7. What is the qualitative difference between the three vortices we have studied? In particular, what happens to a string of dye that is positioned at time zero in each flow along the interval $(-4, -1)$ as times evolves?