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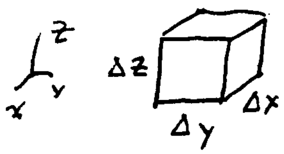
NAVIER STOKES EQUATIONS

1) We want a fluid version of $\vec{F} = m\vec{a}$. To get it we shall consider the forces acting on a small cube of ~~water~~ fluid



. Thus we will be considering Force/unit Vol.

This means we need to replace "m" in $\vec{F} = m\vec{a}$ by the density, ρ, which is mass/unit vol.

2) Recall we found acceleration to be

$$\vec{a} = \frac{D\vec{v}}{Dt} = (\vec{v}\cdot\nabla)\vec{v} + \frac{\partial \vec{v}}{\partial t}$$

where a) $\vec{v} = \langle u(x,y,z,t), v(x,y,z,t), w(x,y,z,t) \rangle$ is the velocity field for the fluid flow. It is what is measured in the Euler approach to fluids. In practice it is measured with anchored buoys in Oceanography and weather stations in meteorology.

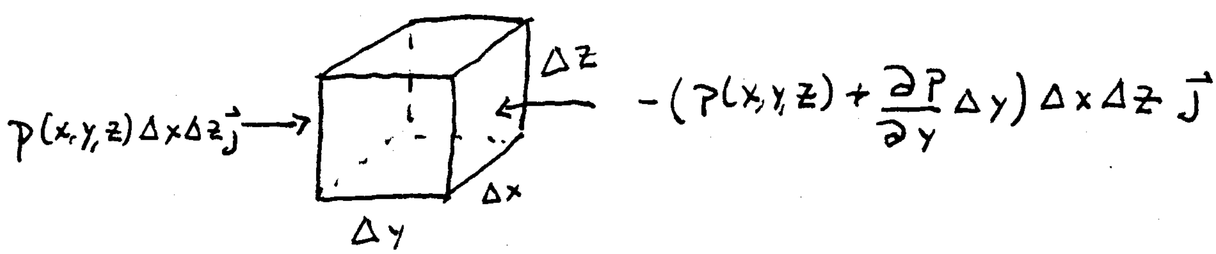
b) $(\vec{v}\cdot\nabla)\vec{v} = \langle \vec{v}\cdot\nabla u, \vec{v}\cdot\nabla v, \vec{v}\cdot\nabla w \rangle$ where

$\nabla u = \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right\rangle$, etc.

3) Thus the $m\vec{a}$ part of $\vec{F} = m\vec{a}$ becomes $\rho \frac{D\vec{V}}{Dt}$.

4) Now, the forces divide into external forces, like gravity and Coriolus, and internal, i.e. forces the surrounding fluid exerts on our sample cube. These internal forces act on the surface and so are F/Area. But their "gradients" are forces/vol., which is what we want.

5) Pressure

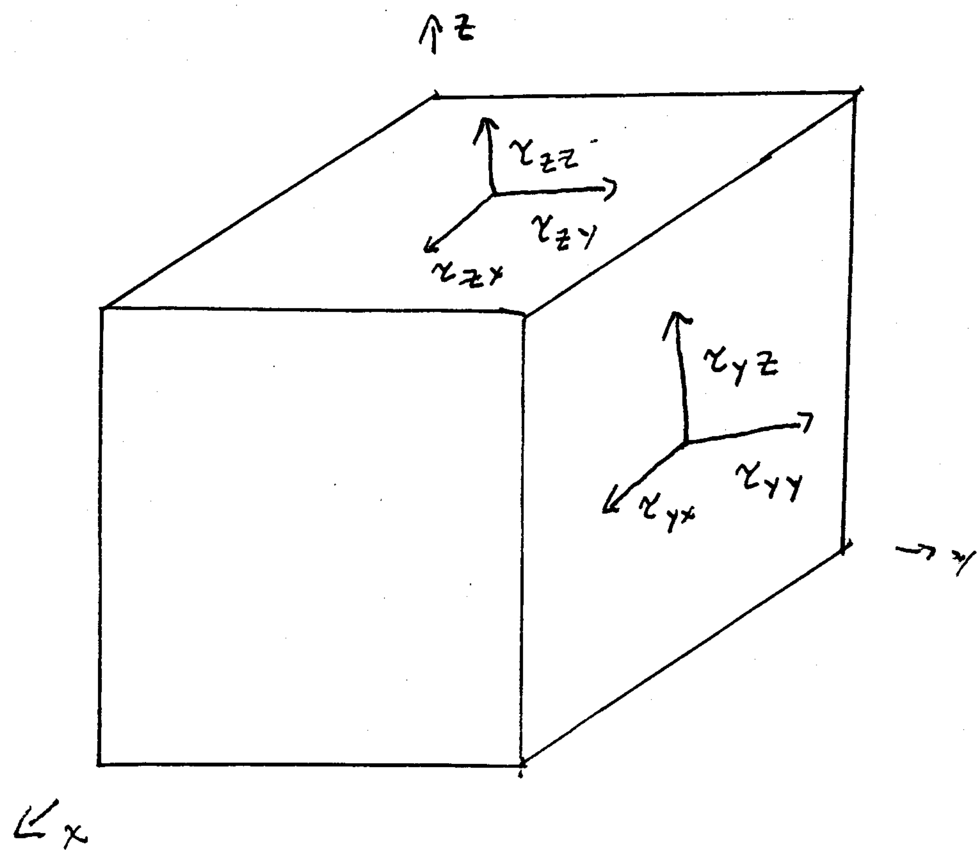


Net force in $\vec{j}$ direction is $-\frac{\partial P}{\partial y} \Delta x \Delta y \Delta z$

So $\vec{F}_{press} = -\nabla p$ where $\vec{F}_{press}$ is force/vol due to pressure.

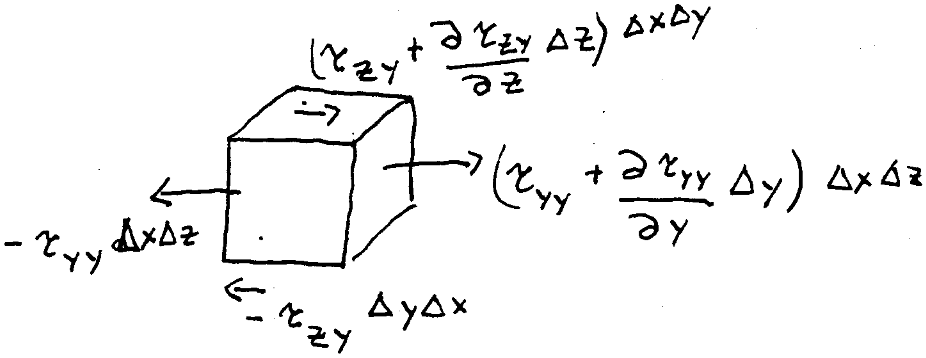
6) Viscosity

STRESS



τ_{ab} = STRESS in b direction on a face normal to a.

It is the difference of these stresses which cause a net force on the body.



So net force in y direction is

$$\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) \Delta V$$

$\frac{\text{Viscous Force}}{\text{Vol.}}$ $\vec{F}_{vis} =$

$$\left\langle \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}, \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}, \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right\rangle$$