

Normal Mode Analysis of the Chesapeake Bay using FEMLAB

Abstract A Normal Mode Analysis (NMA) of the Chesapeake Bay was performed using Dirichlet boundary conditions and FEMLAB. The lowest 100 eigenstates were calculated and compared to a finite difference solution. Based on the normal modes derived numerically, surface current vector fields can be calculated. The vector fields of the Chesapeake Bay provide tools for the solution of problems such as the diffusion of pollutants, particle transport (bio-terrorism), tracking of crab spat, as well as providing a basis set for decomposing real-time currents. Given the difficulty of the boundary, attempts to measure the error in the calculation included tests for orthogonality within the basis set, convergence of the eigenvalue as a function of grid chosen and a comparison to a finite difference calculation for a similar sized grid.

Keywords Normal Mode Analysis · NMA · Chesapeake Bay · FEMLAB 3.0

1 Introduction

The study of computational fluid dynamics has a rich history and the community is still very active. The governing equations, Navier-Stokes, have yet to be analytically solved and the question remains whether the complete solution to fluid flow is even achievable from theoretical grounds. To this end, solutions to flow related problems have necessarily been numerical by nature and solved on computers. The practical goal of achieving near-realtime fluid flows is almost technologically realized through the use of programs such as FEMLAB.

One approach to understanding flow related problems is to employ the same techniques used by the signal processing sector. Given a basis set which fully describes the characteristic modes of a system (the eigenmodes), one

can analyze the state of a system by decomposing the state into amplitudes related to the eigenmodes. The power spectrum achieved for one dimensional time series has proven to be a tremendous tool for many applications. Signal identification and classification represent the simplest of applications. Given a power spectrum, one may speculate that the eigenmodes should evolve in time governed by rules which allows one to forecast signals based on the time evolution of the intensities for each eigenmode. For one dimensional eigenvalue problems, the Fourier series and its associated transform have been the toolset of choice for signal processing for the last fifty years.

With increasing computer power, numerical solutions to the eigenvalue problem in higher dimensions are becoming common-place. The need for computation of eigenmodes is similar to signal processing. When studying a system, having a knowledge of the basis set allows one to describe the state of that system in its most compact language. The numerical aspects of this paper were motivated by a method for completing surface current velocity fields called Normal Mode Analysis (NMA) [Eremeev et al. 1992][1] [2],[Lipphardt et al. 2000][3] .

There have been two studies in recent years [2],[3] which have tested methods for filling in gaps in the velocity field for the data they obtained. Eremeev et al. used the data received from autonomous drifting buoys (ADB) in the Black Sea to extrapolate velocity fields for this closed body of water using what was later termed by Lipphardt's group as Normal Mode Analysis. Eremeev and his collaborators found that this process allowed one to model the large scale currents measured by the ADBs with a relatively small number of modes. One set of data required only 76 modes and yet still accounted for 70% of the kinetic energy associated with the observed field. Lipphardt et al. used this same method to fill in gaps of velocity fields for Monterey Bay. A grid of 39 by 39 was used with HF radar data collected at each point on the grid. This grid spacing corresponds to a minimum spacing between data points of two kilometers. The NMA model used by Eremeev's group did not account for normal flow through an open boundary such as the Monterey Bays boundary with the Pacific Ocean. Lipphardt's group extended the NMA by adding a mode to account for the flow between Mon-

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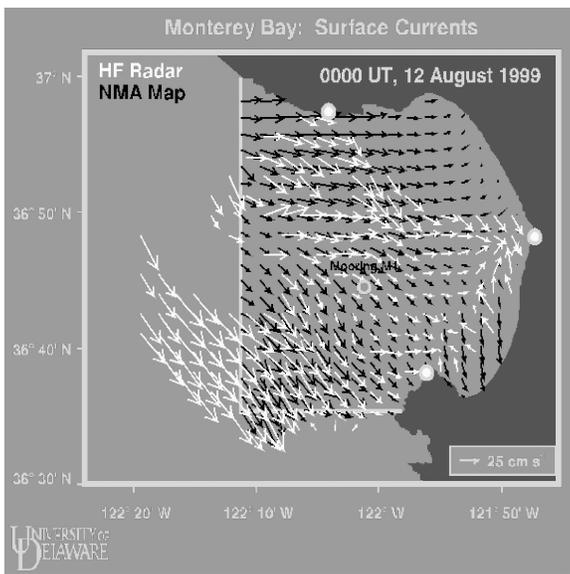


Fig. 1 A "Nowcast" during August 1999, from <http://newark.cms.udel.edu/brucel/slmapping/currents.08.1999.gif>. The white arrows come from HF data and the black arrows are from NMA.

terey Bay and the Pacific. Combining the NMA of the Bay with HF radar observations, they generated velocity fields called "Nowcasts", which can be seen at their website[4]. Figure 1 shows a Nowcast with data taken from HF radar overlaid. Lipphardt and Kirwan used 12 modes to describe the Bay, where each had a kinetic energy equal to at least 15% of the mean. These twelve modes were used to fill out the velocity field obtained by the HF radar [3].

The ability of NMA to efficiently fill in missing current data for coastal regions has many military and civilian applications. As the US Navy begins to increase its focus on the littoral environment, knowledge of surface currents in coastal waterways can significantly improve navigation. Once NMA has been applied to a significant number of waterways, it will become invaluable. The Navy could provide its ships with accurate current data for numerous waterways around the world while only taking data from a small percentage of the total area. This would provide an efficient and cost effective method for providing useful data for both navigation and mission planning.

In the civilian context, NMA can provide a way to help track contaminants and keep tabs on wet life in a region. NMAs ability to complete the picture of the surface current is necessary for accurate computation of packet trajectories for a given waterway. If another Exxon Valdez accident occurred, or chemical or biological agents were introduced into a waterway, a complete picture of the currents in that area would be for the calculation of the dispersion of the contaminants as well as locate the hardest hit areas for both clean up and evacuation. Currents also play an important part in the health of an areas wet life. For the Chesapeake Bay, the waterway with which this project is concerned, NOAA is currently administering a project

to restore the oyster population, which was almost fished to extinction during the 18th and 19th centuries. For this project, current data is important for determining placement of reefs and movement of the oyster population. "If reefs are to be a source of spat for shell plantings, and for sustainability of the reef itself then salinity, flow regime and basin morphology will be important considerations. Hydrodynamic models or drifter studies will be useful in determining the fate of larvae from any proposed reef site" [5].

In this paper, a comparison is made between FEMLAB and a finite difference scheme developed in MATLAB for solving the Laplace type eigenvalue problem with Dirichlet boundary conditions. Given the power and flexibility of FEMLAB, solutions to the surface currents of the Chesapeake Bay were obtained in a fairly straight-forward manner. In order to validate the FEMLAB results, a complementary analysis using finite differences was performed. A number of metrics have been used to compare the output of these two numerical solvers including convergence of eigenvalues, orthogonality within the basis set and the agreement between the two solutions.

2 Normal Mode Analysis

During the past decade a surge of data has become available concerning coastal waterways and estuaries. This influx of data is due to numerous methods of data collection which are being implemented: high frequency radar, Lagrangian drifters, synthetic aperture radar, new generation passive remote-sensing platforms and towed arrays which can collect information on the current in a ships wake [3]. Each of these methods necessarily leaves holes in the data collected due to shadows caused by waves, lack of funding and equipment to complete coverage of the area, or error in the collection and processing of the data. A method called Normal Mode Analysis(NMA) has been developed to fill in the gaps created in these velocity fields and provide a complete picture of the surface currents for a body of water.

NMA separates the spatial and temporal parts of the surface current for a body of water by developing a basis set for the spatial domain. Just like the Galerkin method, the basis set used in NMA already satisfies the boundary conditions for a body of water, zero normal flow at solid boundaries and normal flow consistent with a model or experimental data at open boundaries with other bodies of water. The basis set itself was derived by Eremeev et al. from the formulation developed by Zel'dovich [6] for the description of a three dimensional velocity field using two scalar potentials,

$$\mathbf{u} = \nabla \times [(\hat{n}\Psi) + \nabla \times (\hat{n}\Phi)]. \quad (1)$$

Eremeev and collaborators showed that this formulation can be broken down into two Laplace type eigenvalue

equations for a closed body of water with zero normal flow through the boundaries.

$$\begin{aligned} \nabla^2 \Psi &= -\lambda \Psi && \text{Dirichlet modes} \\ \Psi|_{\partial\Omega} &= 0 \end{aligned} \quad (2)$$

$$\begin{aligned} \nabla^2 \Phi &= -\lambda \Phi && \text{Neumann modes} \\ \partial_n \Phi|_{\partial\Omega} &= 0 \end{aligned} \quad (3)$$

Here Ψ is the stream potential where,

$$(u, v)_D = \left(-\frac{\partial \Psi}{\partial y}, \frac{\partial \Psi}{\partial x} \right), \quad (4)$$

and Φ is the velocity potential where,

$$(u, v)_N = \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y} \right), \quad (5)$$

with (u, v) representing the surface current velocities in the x and y directions respectively. Lipphardt et al. added a third term to account for additional flow through the boundaries with other bodies of water,

$$\begin{aligned} \nabla^2 \Theta(x, y, 0, t) &= S_\Theta(t) && \text{Source terms} \\ (\hat{m} \cdot \vec{\nabla} \Theta)|_{\partial\Omega} &= (\hat{m} \cdot \mathbf{u}_{model})|_{\partial\Omega}. \end{aligned} \quad (6)$$

In equation (6) \hat{m} is the vector normal to the boundary with another body of water and \mathbf{u}_{model} is model or experimental data of the fluid velocity at the boundary. Once all three velocity

3 FEMLAB and Finite Differences

The focus of this paper will be the solution for the stream potentials with Dirichlet boundary conditions for the Chesapeake Bay. The Chesapeake Bay has 11,684 miles of shoreline but is only 189 miles long by 30 miles wide. This provides for an extremely complicated domain forcing the use of a numerical solution. In order to address all three terms listed above, the Dirichlet and Neumann modes are addressed in the standard fashion from within FEMLAB. The source term requires input from an external reference, in this case, the work of Tom Gross at NOAA was employed where source currents were provided from the Quoddy finite element model of the Chesapeake Bay [7]. Although Quoddy is a model of the Chesapeake Bay's currents, it draws data from multiple sets collected near the mouth of the Bay (the Atlantic ocean interface) as well as selected points from the interior of the Bay's geometry. For a recent review of efforts by NOAA and Tom Gross, see reference [7].

While comparing the FEMLAB results with the finite difference results, the same boundary was used, obtained from Quoddy. At its lowest resolution, the default settings from FEMLAB using the Quoddy boundary results in 8735 elements, 1111 boundary elements minimum quality of

Dirichlet Mode #1 - QUODDY Boundary

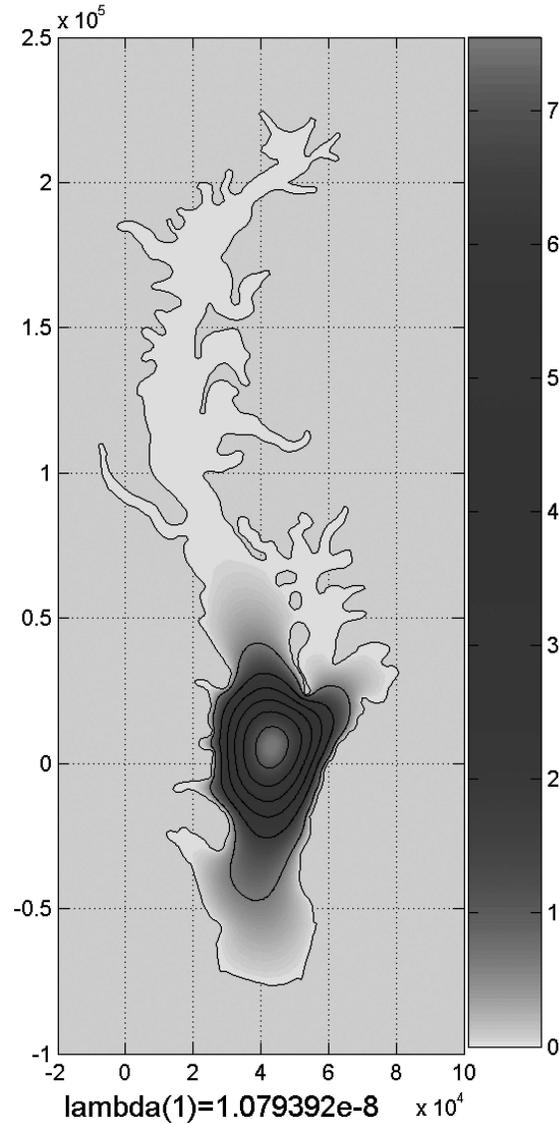


Fig. 2 The lowest normal mode calculated for Dirichlet boundary conditions. The color/intensity represents the height of the scalar potential - which is later transformed into a velocity vector field.

0.5062 and 18579 degrees of freedom. At this minimal grid setting, the lowest 12 modes were solved in 7.4 seconds and all 100 modes were solved in 62 seconds ending with a final error estimated at $1.9e-17$. At this relatively poor resolution, the progression of the eigenvalues obtained appeared consistent until the last 15 values, at which point the eigenvalues began to diverge, a tell-tale sign of approaching the limit of reasonable calculations. At the highest resolution, FEMLAB employed 139760 elements, 4444 boundary elements with a minimum element quality of 0.5062, resulting in 283962 degrees of freedom. On a 2.2 GHz windows computer, the first 12 modes were calculated in just under three minutes. Even with 2 Gbytes

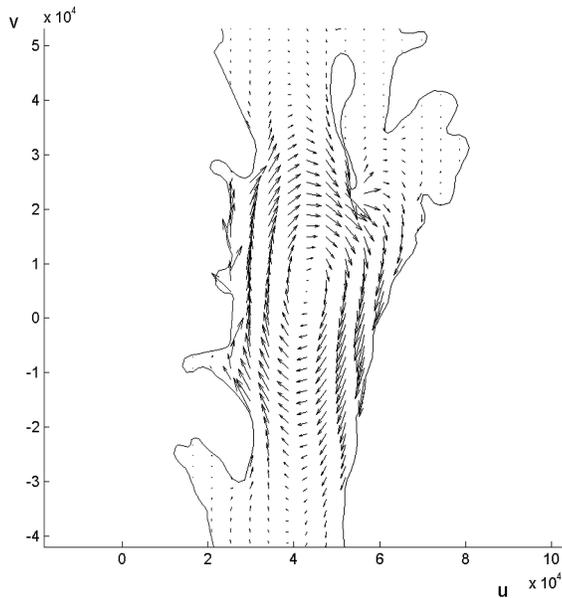


Fig. 3 Velocity vector field $(u, v)_D$ for the lowest Dirichlet mode.

of RAM, evaluating all 100 eigenmodes was not possible. Instead, clusters of 20 eigenstates were calculated to reach the 100th mode. The total time required for 100 modes was approximately 30 minutes of computer time. For the finite difference program a standard central differencing scheme was used,

$$-\lambda \Psi = \frac{-4\Psi + \Psi_{i+1,j} + \Psi_{i-1,j} + \Psi_{i,j+1} + \Psi_{i,j-1}}{h^2}, \quad (7)$$

where the (i, j) notation indicates positions on the grid. Using the Quoddy boundary as a guide, a grid of 350 x 1185 points was created. Boundary conditions were enforced by removing rows and columns in the differentiation matrix corresponding to nodes on the boundary. Calculation of the finite difference scheme took place in MATLAB using *eigs* with considerable pre-processing in order to accommodate the rather large differentiation matrix needed to perform the calculation. As a check, several toy models were created and solved using both FEMLAB and finite differences in order to compare the results against known analytic solutions. This process helped develop confidence in the trends seen as the systems began to approach unreliable behavior.

4 Results and Conclusions

Figure 2 shows the lowest Dirichlet mode calculated using FEMLAB on the Quoddy boundary set at its high resolution. The eigenvalue for this mode is 1.079e-8. On a normalized boundary such as a square, the lowest eigenvalue is typically near 20. The grid for this calculation was not normalized, so there is a factor roughly of $1/(L_x) \times 1/(L_y)$. The boundary size within the Quoddy frame of

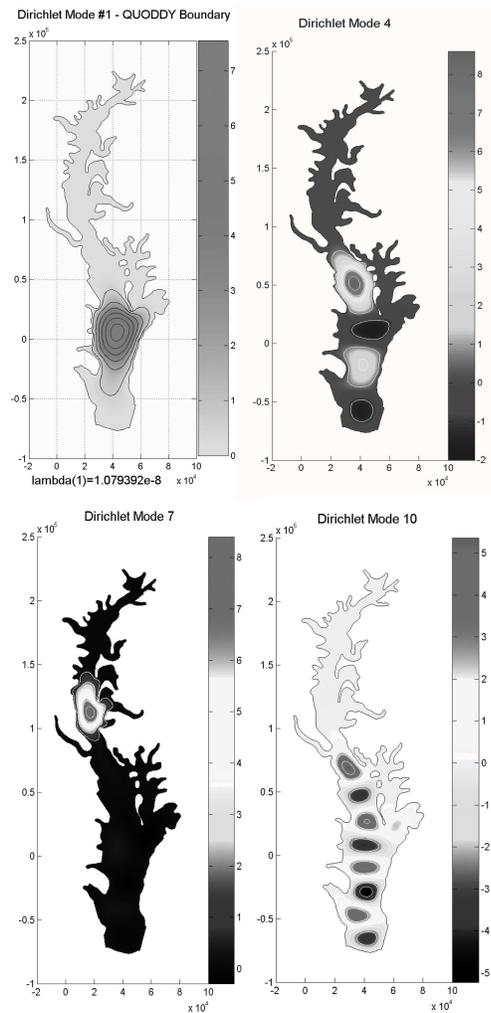


Fig. 4 Normal modes, 1, 4, 7 and 10 are displayed so that the pattern of decreasing wavelength while filling the space of the boundary can clearly be seen. By node 10, the wavelength is too big to fit in the upper Bay.

reference was approximately 10000x25000, which leads to an eigenvalue in the range of 10^{-8} through 10^{-7} . Features of the lowest eigenmode include a central hump filling the largest space within the boundary while maintaining the eigenvalue relationship and meeting the boundary with a value of zero. Bear in mind that this large hump does not represent a displacement of fluid, it is merely an energetic region of space which leads to the calculation of a vorticity flow. This situation is similar to electromagnetism, where the electric and magnetic vector fields can be derived from two scalar potentials. The Dirichlet potential gives rise to vortices, as can be seen in figure 3, which shows the associated vector flow for the lowest eigenmode.

Figures 4 and 5 depict the progression of eigenmodes as the mode number increases from 1 up to 100. In particular, modes 1, 4, 7, 10, 12, 35, 52 and 98 were chosen to

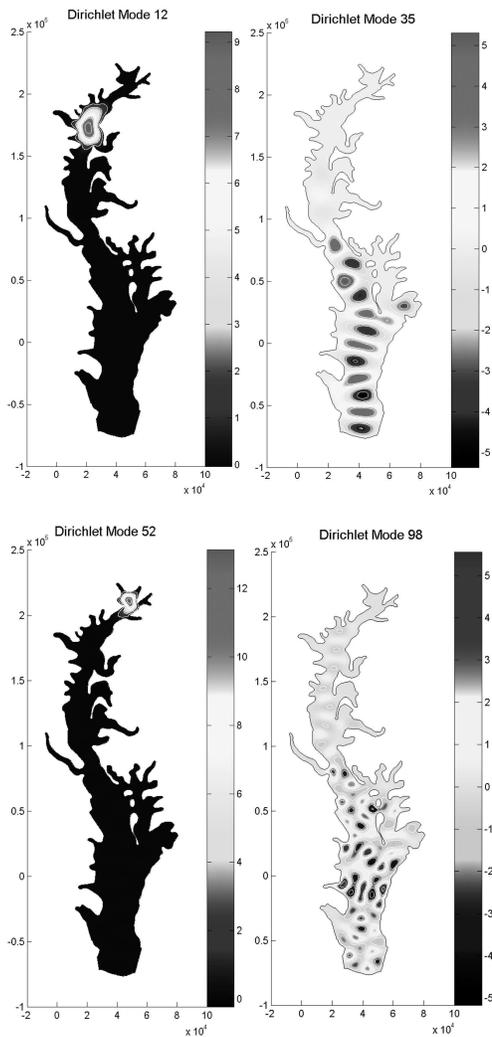


Fig. 5 Normal modes, 12, 36, 52 and 98 are shown. By the 100th mode, every portion of the Bay has been covered at least 10 times. This provides a reasonable set of basis vectors for the Bay. By adding more modes, more coverage of the Bay is possible, leading to more terms used to describe behavior of velocity vector fields. At some higher value, the wavelength becomes so short that variations in the boundary prevent further calculation of modes.

demonstrate the space filling nature of eigenmodes. On a complex boundary, there are occasions when a particular wavelength will fit well within the boundary and as a result, fill most of the domain, which can be seen in modes 4, 10, 35 and 98. Occasionally eigenstates with eigenvalues very close to others which fill the domain will have a wavelength that is not quite right, yet the wavelength might fit exactly between a narrow region of space. Modes 7, 12 and 52 demonstrate this behavior. In total, 100 modes were calculated for the Bay as a design goal so that any one location in the Bay would have had at least 10 modes calculated covering that region of space. With 10 modes covering all regions of the Bay, based on the work of Eremeev et al. and Lipphardt et al., one can confidently pre-

dict that a large fraction of the total energy present in the Bay will have been accounted for using these eigenmodes as a basis set.

To ensure stability in the solution set, several measures were made. Figure 6 shows the relative error for each mode calculated between the FEMLAB result and finite differences. On average, the relative difference in the eigenvalue calculated was three percent. The last five eigenmodes deviated significantly, leading to the suspicion that the solution was diverging. Careful inspection of the behavior from the FEMLAB result indicates that its behavior was smooth from modes 1 through 100. The finite difference results typically varied more than FEMLAB based on toy models, so most likely the error is coming from the finite difference scheme and not FEMLAB. Although not conclusive, the agreement between the two methods indicates stability of the solution set. Other measures, not shown here, include calculation of the deviation of an eigenvalue as the grid resolution was varied. Orthogonality was also used as a measure to ensure the basis set is complete. In both cases, FEMLAB shows superior stability over finite differences.

Although not shown in this report, the Neumann and source terms were calculated for the Chesapeake Bay. Comparison with the finite difference scheme will require further study to validate either the FEMLAB result or the finite differences result. Given that Neumann conditions are better suited for finite element analysis leads one to trust the FEMLAB results, however, improvements to the finite differences are still possible, which will help validate the FEMLAB solution set. The solutions arising from the source term behaved as expected. For a detailed inquiry into the complete eigenmode set calculated to date for the Chesapeake Bay, consult reference [8]. These results can also be seen at the website listed at USNA [9]. Future work on the velocity vector field for the Chesapeake Bay will include full three dimensional analysis using FEMLAB and Quoddy in conjunction.

Concurrent with this study, the residence time of particulates was analyzed using the Navier-Stokes equations and integrating the velocity fields [10]. By comparing trends in velocity fields from an eigenmode calculation to a Navier-Stokes calculation, one can analyze those parts of the velocity domain that give rise to certain behaviors. This comparison represents the initial steps needed to perform a full Normal Mode Analysis. Plans have begun to further instrument the Bay, leading to the question, how many real-time measurements are required in order to predict important behavior within the Chesapeake. Normal Mode Analysis will be used to help answer this question.

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Relative Error of Eigenvalues Between FEMLAB and Finite Difference Scheme

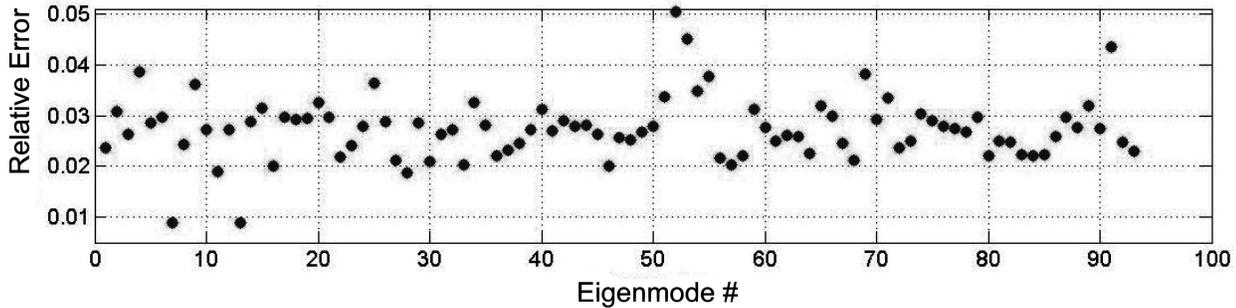


Fig. 6 By comparing the modes calculated from FEMLAB and the finite difference scheme, taking the difference of eigenvalues and dividing by the magnitude gives the relative error. The two schemes agree to within 3 percent on average. Based on experience with known boundaries - where the analytical solution may be employed to check for accuracy, the FEMLAB result is consistently better than the finite difference scheme, leading one to conclude that the 3 percent error is likely to be from finite differences.

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