Bases and Spherical Coordinates

RMM

February 29, 2004

1 Intorduction

Because of the shape of our planet, the natural setting for studying flows in Geophysical Fluid Dynamics is spherical coordinates. Here we begin with the description of these coordinates and develop unit vectors in the directions of the coordinate curves. We then determine the coordinates of a typical vector, such as $\mathbf{\Omega}$, the rotation vectors. Finally we write down the representation of a typical velocity field in spherical coordinates.

2 Coordinate Curves

Let P be a point having coordinates (x, y, z) in **cartesian** coordinates and (ρ, θ, ϕ) in **spherical** coordinates. Here ρ is the distance of P to the origin, θ measures the longitude and ranges between 0 and 2π, and ϕ is the latitude, ranging between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. The cartesian and spherical descriptions of P are related through the following relations:

$$x = \rho\cos\theta\cos\phi, \quad y = \rho\sin\theta\cos\phi, \quad z = \rho\sin\phi. \tag{1}$$

Problem 1: *Show that*

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \text{ArcTan}\frac{y}{x}, \quad \phi = \text{ArcSin}\frac{z}{\sqrt{x^2 + y^2 + z^2}}. \tag{2}$$

Problem 2: *Let P have coordinates $(1, 2, 3)$ in cartesian coordinates. Determine its coordinates in spherical coordinates.*

By a **coordinate curve** in any coordinate system we mean a curve along which only one of the three coordinate parameters varies while the other two are kept constant. For example, the x-axis is a coordinate curve in cartesian coordinates because along this curve x varies while y and z remain constant. Because of the special importance of the three axes in cartesian coordinates, we are interested in identifying the corresponding coordinates curves in spherical coordinates. To that end, let P have coordinates $(\rho_0, \theta_0, \phi_0)$ in spherical coordinate system. Then the coordinate curve one obtains by fixing $\rho = \rho_0$ and $\theta = \theta_0$ while allowing ϕ take on all values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ is a great circle (a **meridian** circle) that passes through P and the two poles. We will refer to this curve as the ϕ-curve through P. Similarly, the coordinate curve one gets by fixing

$\rho = \rho_0$, $\phi = \phi_0$ while allowing θ take on all values between 0 and 2π defines the familiar **parallel** circle through P. We refer to this curve as the θ-curve through P. Finally, fixing $\theta = \theta_0$ and, $\phi = \phi_0$ while allowing ρ take on all values defines the coordinate curve (the ρ-curve) that passes through the origin and P. These three coordinate curves play a role similar to the role that the x, y and z axes play in cartesian coordinates.

Problem 3: *Let $P = (2, 30\ degrees, 60\ degrees)$ in spherical coordinates. Write down the parametrization of the three coordinate curves through P. Plot these curves on the same graph.*

3 Spherical Basis

Given a specific point P on a sphere we now determine three vectors, denoted by $\mathbf{e}_\theta(P)$, $\mathbf{e}_\phi(P)$ and $\mathbf{e}_\rho(P)$. These vectors will play a similar role to $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ of cartesian coordinates in that they will be mutually orthogonal and have length one. By definition, $\mathbf{e}_\theta$ is a unit tangent vector to the θ-curve through P, while e_ϕ is a unit tangent vector to the corresponding ϕ-curve, and $\mathbf{e}_\rho$ is a unit tangent vector to the ρ-curve.

Since by definition $\mathbf{e}_\theta$ is a unit tangent vector to a θ-curve, we begin by parametrizing the θ-curve through the point P. Let P have coordinates $(\rho_0, \theta_0, \phi_0)$ in spherical coordinates. Then the θ-curve through P has the parametrization

$$\mathbf{r}(\theta) = \langle \rho_0 \cos\theta \cos\phi_0, \rho_0 \sin\theta \cos\phi_0, \rho_0 \sin\phi_0 \rangle.$$

To find $\mathbf{e}_\theta$ we differentiate the above expression with respect to θ and divide it by its magnitude to get

$$\mathbf{e}_\theta(P) = -\sin\theta_0 \mathbf{i} + \cos\theta_0 \mathbf{j}. \tag{3}$$

As expected $\mathbf{e}_\theta$ does not have a component in the north-south direction.

Problem 4: Show that $\mathbf{e}_\phi$ and $\mathbf{e}_\rho$ are

$$\mathbf{e}_\phi(P) = -\cos\theta_0 \sin\phi_0 \mathbf{i} - \sin\theta_0 \sin\phi_0 \mathbf{j} + \cos\phi_0 \mathbf{k}. \tag{4}$$

$$\mathbf{e}_\rho(P) = \cos\theta_0 \cos\phi_0 \mathbf{i} + \sin\theta_0 \cos\phi_0 \mathbf{j} + \sin\phi_0 \mathbf{k}. \tag{5}$$

Note that $\mathbf{e}_\phi$, unlike $\mathbf{e}_\theta$, depends on longitude and latitude. Also, as expected, $\mathbf{e}_\rho$ is in the radial direction and is therefore perpendicular to the sphere.

Problem 5: *Show that $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ and $\mathbf{e}_\rho$ are mutually orthogonal.*

Problem 6: *Verify the following relations:*

$$\mathbf{e}_\theta \times \mathbf{e}_\phi = \mathbf{e}_\rho, \quad \mathbf{e}_\phi \times \mathbf{e}_\rho = \mathbf{e}_\theta, \quad \mathbf{e}_\rho \times \mathbf{e}_\theta = \mathbf{e}_\phi. \tag{6}$$

The expressions in (3), (4) and (5) show the relationship between $\{\mathbf{e}_\theta, \mathbf{e}_\phi, \mathbf{e}_\rho\}$ and $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. These relations are invertible. We have

$$\left\{ \begin{array}{ll} \mathbf{i} & = -\sin\theta_0\, \mathbf{e}_\theta - \cos\theta_0 \sin\phi_0\, \mathbf{e}_\phi + \cos\theta_0 \cos\phi_0\, \mathbf{e}_\rho, \\ \mathbf{j} & = \cos\theta_0\, \mathbf{e}_\theta - \sin\theta_0\, \sin\phi_0\, \mathbf{e}_\phi + \sin\theta_0\, \cos\phi_0\, \mathbf{e}_\rho, \\ \mathbf{k} & = \cos\phi_0\, \mathbf{e}_\phi + \sin\phi_0\, \mathbf{e}_\rho. \end{array} \right. \tag{7}$$

Problem 7: Derive (7). Hint: Start with (4) and (5) and eliminate $\mathbf{k}$ between them. Then consider the resulting equation with (3) and solve for $\mathbf{i}$ and $\mathbf{j}$.

What we have accomplished so far is to introduce the concept of *spherical basis vectors* $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ and $\mathbf{e}_\rho$. The significance of this set of mutually orthogonal unit vectors is that any vector $\mathbf{v}$ can be represented in terms of these three vectors as

$$\mathbf{v} = v_1\mathbf{e}_\theta + v_2\mathbf{e}_\phi + v_3\mathbf{e}_\rho. \tag{8}$$

The coefficients v_1, v_2 and v_3 are the coordinates of $\mathbf{v}$ in spherical coordinates.

The same vector $\mathbf{v}$ of course has a representation in terms of the cartesian basis vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$: There are scalars a, b and c such that

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

However, in oceanography and meteorology, it is the spherical representation (8) that is most natural when one studies ocean currents or pressure fronts especially when the study involves large scale structures.

Problem 8: *Use the orthogonality properties of the spherical basis vectors to show that v_1, v_2 and v_3 in (8) are given by*

$$v_1 = \mathbf{v}\cdot\mathbf{e}_\theta, \quad v_2 = \mathbf{v}\cdot\mathbf{e}_\phi, \quad v_3 = \mathbf{v}\cdot\mathbf{e}_\rho \tag{9}$$

Problem 9: *The Earth's rotation vector $\mathbf{\Omega}$ has the form $\Omega\mathbf{k}$ in cartesian coordinates. Find the components of $\mathbf{\Omega}$ in spherical coordinates, i. e., find a, b and c such that*

$$\mathbf{\Omega} = a\,\mathbf{e}_\theta + b\,\mathbf{e}_\phi + c\,\mathbf{e}_\rho.$$

Answer: $a = 0$, $b = \Omega\cos\phi$, $c = \Omega\sin\phi$. Is it intuitively clear why Ω does not have a component in the $\mathbf{e}_\theta$ direction?

4 Acceleration in Spherical Coordinates

We have two options to represent the velocity of a fluid particle: one representation, often called **Lagrangian**, is to write the components of the velocity as functions of t and the position of the particle at t. In other words, one imagines moving with the particle and measuring its velocity as time varies. The second representation of the velocity, called **Eulerian**, represents velocity at a fixed position, that is, one envisions lowering a sensor in a fixed position in a flow and measuring the velocity of the particles that pass through that position as t varies. The Eulerian representation is by far the most common way that fluid flows are modeled.

4.1 Eulerain Formulation of Velocity and Acceleration

Consider a particle of fluid P and its trajectory C consisting of a curve in the three-dimensional space R^3. Let us assume that the position of P at any time t can be specified by a set of differentiable functions $x(t)$, $y(t)$ and $z(t)$ so that

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \tag{10}$$

defines the position vector or the parametrization of the curve C. The velocity $\mathbf{v}$ of P is then determined by direct differentiation of $\mathbf{r}$:

$$\mathbf{v}(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

The above expression defines the Lagrangian representation of the velocity. In fluid dynamics, however, the above components of velocity (i. e., x', y' and z') are converted to functions of position and time, so that typically

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}, \tag{11}$$

where each v_i is a function of position and time:

$$v_i = v_i(x, y, z, t).$$

We refer to (11) as the Eulerian formulation of the velocity field.

The Eulerian representation of velocity has an important impact on computing the acceleration. Using the chain rule of differentiation, the acceleration $\mathbf{a}$ is determined from (11) by the formula

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}. \tag{12}$$

In Cartesian coordinates the components of $\mathbf{a}$ are

$$a_i = \frac{\partial v_i}{\partial t} + \sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial x_j}, \quad i = 1, \ldots, 3. \tag{13}$$

Here we are adopting the convention that $x_1 = x$, $x_2 = y$ and $x_3 = z$. The operator

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{j=1}^{3} v_j \frac{\partial}{\partial x_j} \tag{14}$$

is called the **total** or the **material** derivative. In this notation (14) can be written as

$$a_i = \frac{dv_i}{dt}. \tag{15}$$

4.2 Velocity in Spherical Basis

In order to write down an expression for acceleration in spherical coordinates, we first need to write (14) in spherical coordinates. The position vector $\mathbf{r}(t)$ in (10) takes the form

$$\mathbf{r} = \rho(t)\cos\theta(t)\cos\phi(t)\mathbf{i} + \rho(t)\sin\theta(t)\cos\phi(t)\mathbf{j} + \rho(t)\sin\phi(t)\mathbf{k} \tag{16}$$

in spherical corrdinates. Differentiating (16) with respoect to t yields

$$\begin{aligned}\mathbf{v} = (\rho'\cos\theta\cos\phi - \rho\theta'\sin\theta\cos\phi - \rho\phi'\cos\theta(t)\sin\phi)\mathbf{i}+ \\ (\rho'\sin\theta\cos\phi + \rho\theta'\cos\theta\cos\phi - \rho\phi'\sin\theta\sin\phi)\mathbf{j}+\end{aligned} \tag{17}$$

$$(\rho' \sin\phi + \rho\phi' \cos\phi)\mathbf{k}.$$

Using the formulas (3), (4) and (5) it is easy to see that (17) is equivalent to

$$\mathbf{v} = \rho\theta' \cos\phi \mathbf{e}_\theta + \rho\phi' \mathbf{e}_\phi + \rho' \mathbf{e}_\rho. \tag{18}$$

The coefficients of $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ and $\mathbf{e}_\rho$ in the above expressions are the components of velocity in spherical coordinates. We denote them by v_θ, v_ϕ and v_ρ respectivly, i.e.,

$$\mathbf{v} = v_\theta \mathbf{e}_\theta + v_\phi \mathbf{e}_\phi + v_\rho \mathbf{e}_\rho \tag{19}$$

where

$$v_\theta = \rho\theta' \cos\phi, \quad v_\phi = \rho\phi', \quad v_\rho = \rho' \tag{20}$$

We note that v_θ is the component of the velocity in the east-west direction, v_ϕ is the component in the north-south direction, and v_ρ is the component in the radial direction. We summarize the above discussion in the following theorem.

Theorem 1: When the velocity vector $\mathbf{v}$ is represented in spherical coordinates as in (19) its components v_θ, v_ϕ and v_ρ are related to $\theta(t)$, $\phi(t)$ and $\rho(t)$ through the relations (20).

4.3 Dynamics of Basis Vectors

To compute the acceleration $\mathbf{a}$ in spherical coordinates we need to differentiate (19) with respect to t. Unlike the Cartesian basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ which is independent of t, the spherical basis $\{\mathbf{e}_\theta, \mathbf{e}_\phi, \mathbf{e}_\rho\}$ varies with t because this basis depends on position. Moreover, the particle P, whose acceleration we seek, occupies different positions at different values of t. Going back to (3) we note that

$$\frac{d\mathbf{e}_\theta}{dt} = \mathbf{e}'_\theta = -\theta' \cos\theta \mathbf{i} - \theta' \sin\theta \mathbf{j}.$$

But from (20) we have that $\theta' = \frac{v_\theta}{\rho \cos\phi}$ so the above expression takes the form

$$\frac{d\mathbf{e}_\theta}{dt} = \frac{v_\theta}{\rho \cos\phi}(-\cos\theta \mathbf{i} - \sin\theta \mathbf{j}). \tag{21}$$

Finally, (7) relate the vectors $\mathbf{i}$ and $\mathbf{j}$ to their spherical counterparts. Replacing $\mathbf{i}$ and $\mathbf{j}$ in (21) using the formulas in (7) yields

$$\frac{d\mathbf{e}_\theta}{dt} = \frac{v_\theta}{\rho \cos\phi}(\sin\phi \mathbf{e}_\phi - \cos\phi \mathbf{e}_\rho). \tag{22}$$

Problem 10: Show that $\mathbf{e}_\theta$ and $\frac{d\mathbf{e}_\theta}{dt}$ are orthogonal.

Problem 11: Show that $\frac{d\mathbf{e}_\phi}{dt}$ and $\frac{d\mathbf{e}_\rho}{dt}$ are given by the formulas

$$\frac{d\mathbf{e}_\phi}{dt} = -\frac{v_\theta \tan\phi}{\rho} \mathbf{e}_\theta - \frac{v_\phi}{\rho} \mathbf{e}_\rho, \tag{23}$$

$$\frac{d\mathbf{e}_\rho}{dt} = \frac{v_\theta}{\rho} \mathbf{e}_\theta + \frac{v_\phi}{\rho} \mathbf{e}_\phi. \tag{24}$$

Problem 12: Use the first identity in (6) to prove (24) once (23) and (22) are derived. Hint: Differentiate the first identity in (6) with respect to t.

Problem 13: Show that $\frac{d\mathbf{e}_\phi}{dt}$ is orthogonal to $\mathbf{e}_\phi$ and that $\frac{d\mathbf{e}_\rho}{dt}$ is orthogonal to $\mathbf{e}_\rho$.

4.4 A formula for Acceleration

Going back to (19), we differentiate this relation with respect to t to get

$$\mathbf{a} = \frac{dv_\theta}{dt}\mathbf{e}_\theta + v_\theta\frac{d\mathbf{e}_\theta}{dt} + \frac{dv_\phi}{dt}\mathbf{e}_\phi + v_\phi\frac{d\mathbf{e}_\phi}{dt} + \frac{dv_\rho}{dt}\mathbf{e}_\rho + v_\rho\frac{d\mathbf{e}_\rho}{dt}. \tag{25}$$

Next we substitute (22), (23) and (24) into (25) to get

$$\mathbf{a} = (\frac{dv_\theta}{dt} - \frac{v_\theta v_\phi}{\rho}\tan\phi + \frac{v_\theta v_\rho}{\rho})\mathbf{e}_\theta + (\frac{dv_\phi}{dt} + \frac{v_\theta^2}{\rho}\tan\phi + \frac{v_\phi v_\rho}{\rho})\mathbf{e}_\phi + (\frac{dv_\rho}{dt} - \frac{v_\theta^2 + v_\phi^2}{\rho})\mathbf{e}_\rho. \tag{26}$$

Equation (26) defines acceleration in spherical coordinates.

Problem 14: Verify (26).