Coriolis Force

We saw in the last lecture that

$$2\,\underset{\sim}{\Omega} \times \underset{\sim}{v} \; + \; \underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \underset{\sim}{r})$$

is the contribution of the Earth's rotation to the absolute acceleration of a particle. We also saw that $\|\underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \underset{\sim}{r})\| \leq 0.04\ m/s^2$, a relatively small quantity when compared with $g = 9.8\ m/s^2$. In this note we will take a close look at $2\,\underset{\sim}{\Omega} \times \underset{\sim}{v}$, the <u>Coriolis</u> force. Before doing so, however, we will construct a set of basis vectors in spherical coordinates.

Let (ρ, θ, ϕ) be the spherical coordinates of a point P, where ϕ is the standard latitude. The position vector

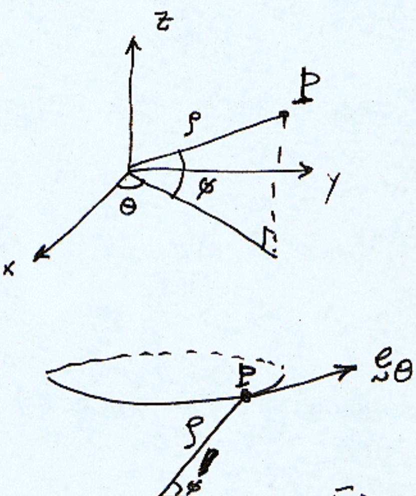


Fig. 1.

(1) $$\underset{\sim}{r} = \langle \rho\cos\theta\cos\phi,\ \rho\sin\theta\cos\phi,\ \rho\sin\phi \rangle$$

defines P. When ρ and ϕ are kept constant and θ is allowed to vary, the point P traces the circle shown in Fig. 1. The parametrization of this curve is (see (1))

$$\underset{\sim}{r}(\theta) = \langle \rho\cos\theta\cos\phi,\ \rho\sin\theta\cos\phi,\ \rho\sin\phi \rangle .$$

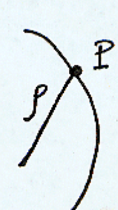
A tangent vector to this curve at P is given by

$$\frac{d\underset{\sim}{r}}{d\theta} = \langle -\rho\sin\theta\cos\phi,\ \rho\cos\theta\cos\phi,\ 0 \rangle$$

whose length, $\left\| \frac{d\underset{\sim}{r}}{d\theta} \right\| = \sqrt{\rho^2\sin^2\theta\cos^2\phi + \rho^2\cos^2\theta\cos^2\phi} = \rho\cos\phi$. So $\underset{\sim}{e}_\theta$, a <u>unit</u> tangent vector to this curve is

$$\boxed{\underset{\sim}{e}_\theta = \langle -\sin\theta,\ \cos\theta,\ 0 \rangle .} \qquad \text{(Eastward unit tangent)}$$

Similarly, consider the curve through P when ρ and θ remain constant (see Fig. 2). Then (see (1) again)



$$\underset{\sim}{r}(\phi) = \langle \rho\cos\theta\cos\phi,\ \rho\sin\theta\cos\phi,\ \rho\sin\phi \rangle$$

is the parametrization of this curve. A tangent vector to this curve at P is derived by differentiating $\underset{\sim}{r}$ with respect to ϕ:

$$\frac{d\underset{\sim}{r}}{d\phi} = \langle -\rho \cos\theta \sin\phi, -\rho \sin\theta \sin\phi, \rho \cos\phi \rangle.$$

Its length, $\left\| \frac{d\underset{\sim}{r}}{d\phi} \right\|$, is

$$\left\| \frac{d\underset{\sim}{r}}{d\phi} \right\| = \sqrt{\underbrace{\rho^2 \cos^2\theta \sin^2\phi + \rho^2 \sin^2\theta \sin^2\phi}_{\rho^2 \sin^2\phi} + \rho^2 \cos^2\phi}$$

$$= \sqrt{\rho^2 \sin^2\phi + \rho^2 \cos^2\phi} = \rho.$$

So, $\underset{\sim}{e}_\phi$, a <u>unit</u> tangent vector to this curve is

$$\boxed{\underset{\sim}{e}_\phi = \langle -\cos\theta \sin\phi, -\sin\theta \sin\phi, \cos\phi \rangle.}$$ (Northward unit vector)

Finally, $\underset{\sim}{r}(\rho) = \langle \rho \cos\theta \cos\phi, \rho \sin\theta \cos\phi, \rho \sin\phi \rangle$, when θ and ϕ are kept constant, defines a curve, a straight line, through P. Its tangent vector is

$$\frac{d\underset{\sim}{r}}{d\rho} = \langle \cos\theta \cos\phi, \sin\theta \cos\phi, \sin\phi \rangle.$$

this vector is already a unit vector. So let

$$\boxed{\underset{\sim}{e}_\rho = \langle \cos\theta \cos\phi, \sin\theta \cos\phi, \sin\phi \rangle}$$ (radial unit vector)

The three vectors $\underset{\sim}{e}_\theta$, $\underset{\sim}{e}_\phi$ and $\underset{\sim}{e}_\rho$ form an <u>orthonormal</u> bases at P.

Let $\underset{\sim}{V}$ be a velocity vector. Then

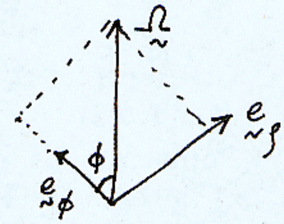
(2) $$\underset{\sim}{V} = u\,\underset{\sim}{e}_\theta + v\,\underset{\sim}{e}_\phi + w\,\underset{\sim}{e}_\rho$$

where u is the component of $\underset{\sim}{V}$ in the eastward direction, v in the northward direction, and w in the radial direction.

We also need to write $\underset{\sim}{\Omega}$ in terms of the basis $\{\underset{\sim}{e}_\theta, \underset{\sim}{e}_\phi, \underset{\sim}{e}_\rho\}$.

Note that $\underset{\sim}{\Omega}$ is in the same plane as $\underset{\sim}{e}_\phi$ and $\underset{\sim}{e}_\rho$.

So $\Omega \cos\phi$ is the component of $\underset{\sim}{\Omega}$ in the direction of $\underset{\sim}{e}_\phi$ and $\Omega \sin\phi$ is the component in the direction of $\underset{\sim}{e}_\rho$, i.e.,



(3) $$\underset{\sim}{\Omega} = \Omega \cos\phi \, \underset{\sim}{e}_\phi + \Omega \sin\phi \, \underset{\sim}{e}_\rho .$$

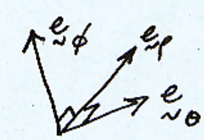
Hence

$$\underset{\sim}{\Omega} \times \underset{\sim}{V} = (\Omega \cos\phi \, \underset{\sim}{e}_\phi + \Omega \sin\phi \, \underset{\sim}{e}_\rho) \times (u \underset{\sim}{e}_\theta + v \underset{\sim}{e}_\phi + w \underset{\sim}{e}_\rho)$$

$$= \Omega u \cos\phi \, \underset{\sim}{e}_\phi \times \underset{\sim}{e}_\theta + \Omega w \cos\phi \, \underset{\sim}{e}_\phi \times \underset{\sim}{e}_\rho + \Omega u \sin\phi \, \underset{\sim}{e}_\rho \times \underset{\sim}{e}_\theta + \Omega v \sin\phi \, \underset{\sim}{e}_\rho \times \underset{\sim}{e}_\phi .$$

But $\underset{\sim}{e}_\phi \times \underset{\sim}{e}_\theta = -\underset{\sim}{e}_\rho$, $\underset{\sim}{e}_\phi \times \underset{\sim}{e}_\rho = \underset{\sim}{e}_\theta$,

$\underset{\sim}{e}_\rho \times \underset{\sim}{e}_\theta = \underset{\sim}{e}_\phi$, $\underset{\sim}{e}_\rho \times \underset{\sim}{e}_\phi = -\underset{\sim}{e}_\theta$.



So $$\underset{\sim}{\Omega} \times \underset{\sim}{V} = -\Omega u \cos\phi \, \underset{\sim}{e}_\rho + \Omega w \cos\phi \, \underset{\sim}{e}_\theta + \Omega u \sin\phi \, \underset{\sim}{e}_\phi - \Omega v \sin\phi \, \underset{\sim}{e}_\theta$$

$$= (\Omega w \cos\phi - \Omega v \sin\phi) \underset{\sim}{e}_\theta + \Omega u \sin\phi \, \underset{\sim}{e}_\phi - \Omega u \cos\phi \, \underset{\sim}{e}_\rho$$

$$= \langle \Omega w \cos\phi - \Omega v \sin\phi, \ \Omega u \sin\phi, \ -\Omega u \cos\phi \rangle .$$

In many problem we study, $w \approx 0$ and the third component of the Coriolis force, $-2\Omega u \cos\phi$, is neglected. Let $\boxed{f = 2\Omega \sin\phi}$. f is called the Coriolis parameter. Then approximately

$$\boxed{2\,\Omega \times V = \langle \ fv, \ fu, \ 0 \rangle}$$