Application of the Finite Difference Method to Normal Modes

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In this note we apply the standard finite difference method to compute normal modes of the Dirichlet boundary value problem

$$-\Delta\psi = \lambda\psi, \qquad \mathbf{x} \in \Omega, \tag{1}$$

where $\Delta = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}$ is the Laplace operator, subject to

$$\psi|_{\partial\Omega} = 0, \tag{2}$$

where Ω is a two-dimensional domain and $\partial\Omega$ is its boundary.

The finite difference method replaces derivatives by their respective finite differences. To motivate this idea recall the Tyalor series formula for a function of a single variable:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(iv)}(x) + \dots \tag{3}$$

We also recall the definition of $f'(a)$:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}. \tag{4}$$

We can therefore approximate $f'(a)$ by

$$\frac{f(a+h) - f(a)}{h} \tag{5}$$

which, using (3), results in an error of the size

$$\frac{h}{2}f''(a) + \dots \tag{6}$$

where the dots in the above expression contain terms of order h^2 or higher. We refer to this term as $o(h)$ and say that (5) is an $o(h)$ approximation of $f'(a)$.

Our main task is to replace the Laplace operator Δ with an appropriate finite difference approximation. First we learn how to replace any second derivative with its finite difference. To that end we consider the following two Taylor series (the first is (3); the second we get from (3) by replacing h with $-h$):

$$\begin{cases} f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(iv)}(x) + \dots, \\ f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(iv)}(x) + \dots. \end{cases} \tag{7}$$

Add the two equations and solve for $f''(x)$ to get

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + \frac{h^2}{12} f^{(iv)}(x) + \dots . \tag{8}$$

The above expression is called the **centered finite difference** approximation of $f''(x)$. Note that this formula gives us an $o(h^2)$ approximation of $f''(x)$.

We are now ready to implement the finite difference method to (1). Let (x_i, y_j) be a rectangular grid laid out to cover Ω. Let

$$\psi_{i,j} \tag{9}$$

denote $\psi(x_i, y_j)$. Using (8) we replace $\Delta\psi(x_i, y_j)$ by its centered finite difference approximation:

$$\frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{h^2} + \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{k^2}, \tag{10}$$

where k is the stepsize in the y-direction (in most applications $k = h$.). We now substitute (10) into (1) to get

$$\frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{h^2} + \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{k^2} = -\lambda\psi_{i,j} \tag{11}$$

where i and j range over all the points in the grid. We have thus replaced the original eigenvalue-eigenfunction problem (1) with a set of algebraic equations for $\psi_{i,j}$. We note the boundary condition (2) is automatically embedded in (11).

The set of equations (11) can be viewed in the matrix form

$$A\mathbf{x} = \mathbf{0} \tag{12}$$

where $\mathbf{x}$ is the vector of $\psi_{i,j}$s and A is the associated companion square matrix of coefficients in (11). We emphasize that λ, the eigenvalue in (1), is present in A. System (12) typically has the trivial vetor $\mathbf{x} = \mathbf{0}$ as its unique solution unless A is a singular matrix. So we now choose λ so that A is a singular matrix. The (nonunique) solution to (12) is then the eigenfunction ψ we seek. This is the fundamental idea behind using the finite difference method to compute normal modes of the Laplace operator.

As our first example belwo will demonstarte, the matrix A is quite sparse. So this formulation is ideal for using numerical linear algebra techniques that are already developed in MATLAB, as well as for using parallel algorithms developed for a Beowulf cluster system.

Example 1

Let $\Omega = (0,1) \times (0,1)$. With this choice of Ω the normal mode problem (1) has the **exact solution**

$$\psi_{m,n}(x,y) = A_{m,n} \sin n\pi x \sin m\pi y, \quad \lambda_{m,n} = (n^2 + m^2)\pi^2. \tag{13}$$

We apply the finite difference method to (1) with this choise of Ω and try to approximate as many eigenvalues and eigenfunctions in (13) as possible.

We construct a rectangular grid in Ω be selecting an integer n and subdividing $(0,1)$ into $n+1$ subintervals of equal length $\frac{1}{n+1}$. Then $x_i = \frac{i}{n+1}$, $y_j = \frac{j}{n+1}$ for some n. So $h = k = \frac{1}{n+1}$ and i and j range from 1 to n while $i = 0$, $i = n+1$, $j = 0$ and $j = n+1$ define the boundaries of the square. Therefore $\psi 0, j = \psi)n+1, j = \psi_{i,0} = \psi_{i,n+1} = 0$ for i and j. Some of the typical equations in (11) look as follows:

$$\begin{array}{rcl}
i=1, j=1 & : & \psi_{2,1} + \psi_{1,2} + (\lambda h^2 - 4)\psi_{1,1} = 0, \\
i=2, j=1 & : & \psi_{3,1} + \psi_{1,1} + \psi_{2,2} + (\lambda h^2 - 4)\psi_{2,1} = 0. \\
i=3, j=1 & : & \psi_{4,1} + \psi_{3,1} + \psi_{3,2} + (\lambda h^2 - 4)\psi_{3,1} = 0. \\
3 < i < n, j=1 & : & \ldots \quad \ldots \quad \ldots = 0. \\
i=n, j=1 & : & \psi_{n-1,1} + \psi_{n,2} + (\lambda h^2 - 4)\psi_{n,1} = 0. \\
i=1, j=2 & : & \psi_{2,2} + \psi_{1,3} + (\lambda h^2 - 4)\psi_{1,2} = 0. \\
i=2, j=2 & : & \psi_{3,2} + \psi_{1,2} + \psi_{2,3} + \psi_{2,1} + (\lambda h^2 - 4)\psi_{2,2} = 0. \\
3 < i < n, j=2 & : & \ldots \quad \ldots \quad \ldots = 0. \\
i=n, j=2 & : & \psi_{n-1,2} + \psi_{n,3} + \psi_{n,1} + (\lambda h^2 - 4)\psi_{n,2} = 0.
\end{array} \tag{14}$$

The above set of equation can now be transformed in the matrix form (12). Here the vector $\mathbf{x}$ and the matrix A takes the form

$$\mathbf{x} = \begin{bmatrix} \psi_{1,1} \\ \psi_{2,1} \\ \psi_{3,1} \\ \cdots \\ \cdots \\ \psi_{n,1} \\ \psi_{1,2} \\ \psi_{2,2} \\ \psi_{3,2} \\ \cdots \\ \cdots \\ \psi_{n,2} \\ \cdots \\ \cdots \\ \psi_{1,n} \\ \psi_{2,n} \\ \psi_{3,n} \\ \cdots \\ \cdots \\ \psi_{n,n} \end{bmatrix}. \tag{15}$$

$$A = \begin{bmatrix} \lambda h^2 - 4 & 1 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \cdots \\ 1 & \lambda h^2 - 4 & 1 & 0 & 0 & \cdots & 1 & 0 & 0 \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{16}$$

The matrix A has the value $\lambda h^2 - 4$ on its diagonal, the value 1 on the two subdiagonals immediately adjacent to the main diagonal, and two other subdiagonals consisting of

1's. All other entries of A are zero. Hence, when n is large the matrix A is quite sparse and we will use special techniques from computational matrix theory to determine λ so that its determinant vanishes.