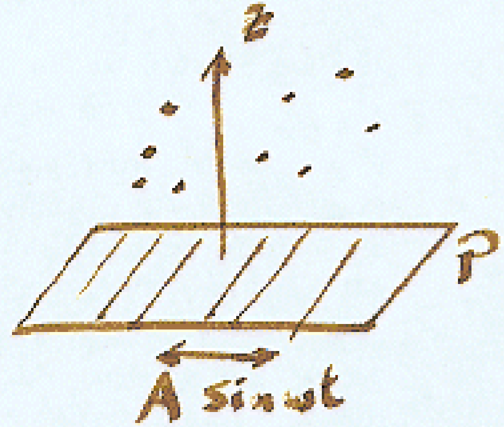
Flow over an Oscillating Plate



The plate P oscillates back and forth according to the formula A sin ωt. The problem is to find the velocity field of typical particles above the plate. Equations of motion are:

$$\left\{\begin{array}{l}
\rho\left(\dfrac{\partial v_1}{\partial t} + v_1\dfrac{\partial v_1}{\partial x} + v_2\dfrac{\partial v_1}{\partial y} + v_3\dfrac{\partial v_1}{\partial z}\right) = -\dfrac{\partial p}{\partial x} + \mu\Delta v_1, \\[2ex]
\rho\left(\dfrac{\partial v_2}{\partial t} + v_1\dfrac{\partial v_2}{\partial x} + v_2\dfrac{\partial v_2}{\partial y} + v_3\dfrac{\partial v_2}{\partial z}\right) = -\dfrac{\partial p}{\partial y} + \mu\Delta v_2, \\[2ex]
\rho\left(\dfrac{\partial v_3}{\partial t} + v_1\dfrac{\partial v_3}{\partial x} + v_2\dfrac{\partial v_3}{\partial y} + v_3\dfrac{\partial v_3}{\partial y}\right) = -\dfrac{\partial p}{\partial z} + \mu\Delta v_3 - \rho g, \\[2ex]
\dfrac{\partial v_1}{\partial x} + \dfrac{\partial v_2}{\partial y} + \dfrac{\partial v_3}{\partial z} = 0
\end{array}\right.$$

Ansatz (Form of solution):

$$v_1 = u(z,t), \quad v_2 = 0, \quad v_3 = 0.$$

(2)

Boundary conditions:

$$u(0,t) = A \sin \omega t, \qquad \lim_{z \to \infty} u(z,t) = 0.$$

Solution Method: (Separation of Variables)

Because $u(0,t)$ depends on $\sin \omega t$, we expect that $u(z,t)$ will depend on $\sin \omega t$ or $\cos \omega t$. Since $e^{i\omega t} = \cos \omega t + i \sin \omega t$, we choose $u(z,t)$ as

$$u(z,t) = F(y) e^{i\omega t}.$$

Substitute the ansatz into the equations of motion: The four equations reduce to (assume $P = P(z)$ only)

$$\begin{cases} \rho \dfrac{\partial u}{\partial t} = \mu \dfrac{\partial^2 u}{\partial z^2} \\[2ex] 0 = -\dfrac{\partial P}{\partial z} - \rho g. \end{cases}$$

Integrate the second equation with respect to z to get

$$P(z) = -\rho g z + P_0$$

where P_0 is the pressure at $z = 0$. Once z is large enough so that P becomes negative, this model loses its significance.

③ Substitute $u = Fe^{i\omega t}$ into $\rho u_t = \mu u_{zz}$:

$$\rho (i\omega) e^{i\omega t} F = \mu F'' e^{i\omega t}, \text{ or}$$

$$F'' - ik^2 F = 0 \quad , \text{ where } \quad k^2 = \frac{\rho\omega}{\mu}.$$

Seek solutions of the form

$$F(z) = e^{mz}$$

for $F'' - ik^2 F = 0$. m satisfies

$$m^2 = ik^2.$$

or $\quad m = \pm k\sqrt{i}$. But $\sqrt{i} = \pm \frac{1}{\sqrt{2}}(1+i)$ (how do you prove this fact?) so

$$F(z) = e^{\pm \frac{k}{\sqrt{2}}(1+i)z}.$$

One of our boundary conditions states that $\lim\limits_{z\to\infty} u = 0$, so discard the + solution is F. So

$$F(z) = e^{-\frac{k}{\sqrt{2}}(1+i)z} \text{ and}$$

$$u = e^{i\omega t} e^{-k/\sqrt{2}(1+i)z}$$

or

$$u = e^{-\frac{k}{\sqrt{2}}z} e^{i(\omega t - \frac{k}{\sqrt{2}}z)}$$

(4) After using Euler's formula ($e^{ix} = \cos x + i\sin x$), we get

$$u = e^{-\frac{k}{\sqrt{2}}z}\left(\cos\left(\omega t - \frac{k}{\sqrt{2}}z\right) + i\sin\left(\omega t - \frac{k}{\sqrt{2}}z\right)\right).$$

So the real-valued solution u is

$$u = e^{-\frac{k}{\sqrt{2}}z}\left(c_1\cos\left(\omega t - \frac{k}{\sqrt{2}}z\right) + c_2\sin\left(\omega t - \frac{k}{\sqrt{2}}z\right)\right)$$

But $u(0,t) = A\sin\omega t$, according to the first boundary condition. So

$$A\sin\omega t = c_1\cos\omega t + c_2\sin\omega t$$

which implies that

$$c_1 = 0, \quad c_2 = A.$$

So

$$u = Ae^{-\frac{k}{\sqrt{2}}z}\sin\left(\omega t - \frac{k}{\sqrt{2}}z\right)$$

where $k^2 = \dfrac{\rho\omega}{\mu}$.