

INDEPENDENCE SEQUENCES OF WELL-COVERED GRAPHS: NON-UNIMODALITY AND THE ROLLER-COASTER CONJECTURE

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ABSTRACT. A graph G is well-covered provided each maximal independent set of vertices has the same cardinality. The term s_k of the independence sequence $(s_0, s_1, \dots, s_\alpha)$ equals the number of independent k -sets of vertices of G . We investigate constraints on the linear orderings of the terms of the independence sequence of well-covered graphs. In particular, we provide a counterexample to the recent unimodality conjecture of Brown, Dilcher, and Nowakowski. We formulate the Roller-Coaster Conjecture to describe the possible linear orderings of terms of the independence sequence.

1. INTRODUCTION

This note investigates the independence sequences of well-covered graphs. We provide a counterexample to a recent conjecture on the unimodality of such sequences, and we formulate the Roller-Coaster Conjecture to describe the possible linear orderings of terms of the independence sequence.

Let $G = (V, E)$ be an undirected graph without loops or multiple edges. A set V' of vertices is *independent* or *stable* provided no two vertices in V' are joined by an edge of the graph G . We say V' is an *independent k -set* provided V' has cardinality k . The *independence number* of G is the maximum cardinality of an independent set and is denoted by $\alpha = \alpha(G)$. With the graph G we associate the *independence sequence*

$$(s_0, s_1, \dots, s_\alpha),$$

where s_k equals the number of independent k -sets of vertices in G ($k = 0, 1, \dots, \alpha$). Note that $s_0 = 1$ (the empty set is independent), and $s_1 = |V|$. The *independence polynomial*

$$S(G, z) = s_0 + s_1z + s_2z^2 + \dots + s_\alpha z^\alpha$$

of G is the generating function for the sequence $(s_0, s_1, \dots, s_\alpha)$.

Examples. (a) The complete graph K_b on b vertices has $\alpha = 1$ and independence polynomial $S(K_b, z) = 1 + bz$.

(b) More generally, in the disjoint union αK_b of α copies of K_b there are $s_k = b^k \binom{\alpha}{k}$ independent k -sets for $k = 0, 1, \dots, \alpha$. The independence polynomial is

$$S(\alpha K_b, z) = (1 + bz)^\alpha.$$

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Alavi, Malde, Schwenk and Erdős [1] investigated the possible orderings of the independence numbers $s_1, s_2, \dots, s_\alpha$ of a graph. In 1987 they showed that independence sequences of graphs are unconstrained with respect to order in the following strong sense:

Proposition 1 (Alavi *et al*). *For any permutation π of the set $\{1, 2, \dots, \alpha\}$ there exists a graph whose independence sequence $(s_0, s_1, \dots, s_\alpha)$ satisfies*

$$(1) \quad s_{\pi(1)} < s_{\pi(2)} < \dots < s_{\pi(\alpha)}.$$

Brown, Dilcher, and Nowakowski [3] recently considered a similar question for the class of well-covered graphs. A graph G is *well-covered* provided every maximal independent set has the same cardinality α . Well-covered graphs have been actively studied since their formal introduction by Plummer [9] in 1970. The graphs in Examples (a) and (b) are well-covered, and their independence sequences are *unimodal*, that is, there exists an index p such that

$$s_0 \leq s_1 \leq \dots \leq s_p \quad \text{and} \quad s_p \geq s_{p+1} \geq \dots \geq s_\alpha.$$

The motivation of this note is the Unimodality Conjecture of Brown, Dilcher, and Nowakowski [3]:

The Unimodality Conjecture. *Well-covered graphs have unimodal independence sequences.*

The Unimodality Conjecture is trivial for graphs with independence number $\alpha \in \{1, 2\}$ and is verified below for $\alpha = 3$. However, we provide counterexamples for $\alpha \in \{4, 5, 6, 7\}$. We propose a new conjecture, the Roller-Coaster Conjecture, which asserts that the independence numbers $s_{\lceil \alpha/2 \rceil}, s_{\lceil \alpha/2 \rceil + 1}, \dots, s_\alpha$ of a well-covered graph are actually *unconstrained* in the sense of Alavi *et al*. We verify our conjecture for independence numbers $\alpha \leq 7$.

Our work may be viewed as a first step toward a solution of the following difficult problem [2]:

Problem 2. *Find necessary and sufficient conditions for a sequence $(s_0, s_1, \dots, s_\alpha)$ to be the independence sequence of a well-covered graph.*

Independence sequences of graphs have received attention in the combinatorial literature under two alternative guises. First, the *clique polynomial* of a graph G is just the independence polynomial of the complement of G . (References on clique polynomials include [4, 5].) Second, with each graph G we may associate a simplicial complex, whose faces are the independent sets of G . Well-covered graphs correspond to *equidimensional flag complexes* [10], and the independence numbers s_j of a well-covered graph are related to the terms of the corresponding f -vector $(f_{-1}, f_0, \dots, f_{\alpha-1})$ by the index shift $s_j = f_{j-1}$. (See [10] and Section 4 of [11] for recent work on f -vectors of flag complexes and connections to independence polynomials.) The powerful Kruskal-Katona Theorem [7, 8] characterizes f -vectors of simplicial complexes; however, our restriction to the class of well-covered graphs renders such a characterization more difficult. We shall focus on the more tractable problem of obtaining results on possible orderings among terms of the independence sequence in the spirit of Alavi *et al*.

2. THE ROLLER-COASTER CONJECTURE

We begin by showing that there are indeed some constraints on independence sequences of well-covered graphs.

Theorem 3. *The independence sequence $(s_0, s_1, \dots, s_\alpha)$ of a well-covered graph G satisfies*

$$(2) \quad \frac{s_0}{\binom{\alpha}{0}} \leq \frac{s_1}{\binom{\alpha}{1}} \leq \dots \leq \frac{s_\alpha}{\binom{\alpha}{\alpha}}.$$

Proof. We employ a flag-counting argument. Let \mathcal{S}_k denote the set of independent k -sets of vertices in G ($k = 0, \dots, \alpha$). We consider the cardinality of the set

$$F_k = \{(V_k, V_{k+1}) : V_k \subset V_{k+1}, V_k \in \mathcal{S}_k, V_{k+1} \in \mathcal{S}_{k+1}\} \quad (k = 0, \dots, \alpha - 1).$$

On the one hand, each V_{k+1} contains exactly $k + 1$ subsets V_k of cardinality k . Thus $|F_k| = (k + 1)s_{k+1}$. On the other hand, because G is well-covered, each V_k is contained in at least one independent α -set and hence in at least $\alpha - k$ independent sets of cardinality $k + 1$. Thus $|F_k| \geq (\alpha - k)s_k$. Therefore $(\alpha - k)s_k \leq (k + 1)s_{k+1}$, from which (2) follows. \square

Remarks.

- (a) The inequalities in (2) are sharp for $G = \alpha K_1$.
- (b) Theorem 3 implies that the Unimodality Conjecture is true for $\alpha = 3$.
- (c) From (2) one easily deduces the inequalities in the following corollary, which can also be deduced from general results of Hibi [6] on equidimensional multicomplexes. Also see Problem 2 on page 135 of Stanley's book [10].

Corollary 4. *Let $(s_0, s_1, \dots, s_\alpha)$ be the independence sequence of a well-covered graph. Then $s_i \leq s_j \leq s_{\alpha-i}$ for $i \leq j \leq \alpha - i$. In particular, $s_0 \leq s_1 \leq \dots \leq s_{\lceil \alpha/2 \rceil}$.*

We know of no constraints for independence sequences of well-covered graphs, other than those imposed by Theorem 3. We propose a conjecture, whose name derives from the schematic in Figure 1.

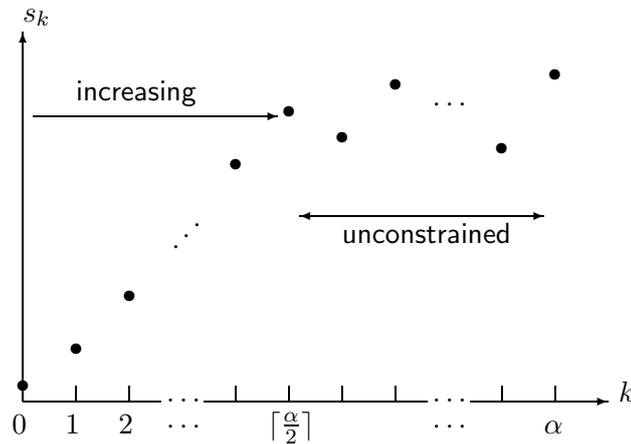


FIGURE 1. The independence sequence $(s_0, s_1, \dots, s_\alpha)$ of a well-covered graph satisfies $s_0 \leq s_1 \leq \dots \leq s_{\lceil \alpha/2 \rceil}$ by Theorem 3. The Roller-Coaster Conjecture asserts that the terms $s_{\lceil \alpha/2 \rceil}, \dots, s_\alpha$ are unconstrained with respect to order.

Roller-Coaster Conjecture. For any permutation π of the set $\{\lceil \alpha/2 \rceil, \dots, \alpha\}$ there exists a well-covered graph whose independence sequence $(s_0, s_1, \dots, s_\alpha)$ satisfies

$$s_{\pi(\lceil \alpha/2 \rceil)} < s_{\pi(\lceil \alpha/2 \rceil + 1)} < \dots < s_{\pi(\alpha)}.$$

The Roller-Coaster Conjecture is readily verified for $\alpha \in \{1, 2, 3\}$. In the next section we verify this conjecture for all graphs with independence number $\alpha = 4$ and $\alpha = 6$. In Section 4 we extend our analysis to handle the cases $\alpha = 5$ and $\alpha = 7$.

3. A SPECIAL CASE: FLAT ROLLER-COASTERS

In this section we establish the Roller-Coaster Conjecture for graphs with independence numbers 4 and 6. We first produce a *flat roller-coaster graph*, that is, a graph with $s_{\lceil \alpha/2 \rceil} = s_{\lceil \alpha/2 \rceil + 1} = \dots = s_\alpha$; we then perturb the graph to achieve all linear orderings of $s_{\lceil \alpha/2 \rceil}, s_{\lceil \alpha/2 \rceil + 1}, \dots, s_\alpha$.

Let G_1 and G_2 be vertex-disjoint graphs. Recall that the *join* $G_1 + G_2$ is the graph obtained from disjoint copies of G_1 and G_2 by inserting all possible edges joining vertices in G_1 and G_2 . The join of two well-covered graphs with the same independence number α is also a well-covered graph with independence number α . The independence polynomial for the join of two graphs is readily found:

$$(3) \quad S(G_1 + G_2, z) = S(G_1, z) + S(G_2, z) - 1.$$

The subtraction of 1 ensures that the empty set is accounted for properly. (The join of two graphs is the complement of the disjoint union of their complements, and thus had we worked with clique polynomials in place of independence polynomials, our constructions below would involve disjoint unions of complete multipartite graphs.)

Now we treat the case $\alpha = 4$ of the Roller-Coaster Conjecture. We construct a flat roller-coaster graph by joining copies of the graphs αK_j from Example (b): let

$$(4) \quad G = G_1 + G_4 + G_{10}, \quad \text{where } G_j = \underbrace{(4K_j + \dots + 4K_j)}_{w_j} \quad (j = 1, 4, 10).$$

The non-negative integers w_1, w_4 , and w_{10} in (4) will be chosen soon. First observe that G is well-covered with independence number $\alpha = 4$. Moreover, from formula (3) and Example (b) we see that the independence polynomial of G is

$$\begin{aligned} S(G, z) &= w_1(1+z)^4 + w_4(1+4z)^4 + w_{10}(1+10z)^4 - (w_1 + w_4 + w_{10} - 1) \\ &= s_0 + s_1z + s_2z^2 + s_3z^3 + s_4z^4. \end{aligned}$$

The terms s_2, s_3 , and s_4 of the independence sequence are related to the parameters w_1, w_4 , and w_{10} by the linear system

$$(5) \quad \begin{bmatrix} s_2 \\ s_3 \\ s_4 \end{bmatrix} = A \begin{bmatrix} w_1 \\ w_4 \\ w_{10} \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} 6 \cdot 1^2 & 6 \cdot 4^2 & 6 \cdot 10^2 \\ 4 \cdot 1^3 & 4 \cdot 4^3 & 4 \cdot 10^3 \\ 1 \cdot 1^4 & 1 \cdot 4^4 & 1 \cdot 10^4 \end{bmatrix}.$$

In particular, when $\mathbf{w} = (w_1, w_4, w_{10})^T = (8000, 15, 4)^T$, then $s_2 = s_3 = s_4 = 51840$, and G is a flat roller-coaster graph.

To obtain a well-covered graph G with a prescribed ordering of s_2, s_3 , and s_4 , we first choose $\mathbf{r} = (r_2, r_3, r_4) \in \mathbb{Q}^3$ so that the components of $\mathbf{s} = (51840 + r_2, 51840 + r_3, 51840 + r_4)$ satisfy the ordering. Because \mathbf{w} depends continuously on \mathbf{s} , and because

$$\begin{bmatrix} 8000 \\ 15 \\ 4 \end{bmatrix} = A^{-1} \begin{bmatrix} 51840 \\ 51840 \\ 51840 \end{bmatrix}$$

is a vector with positive components, we can rescale \mathbf{r} so that $\mathbf{w} = (w_1, w_4, w_{10})^T = A^{-1}\mathbf{s}$ is a vector of positive rationals. Clearing denominators by multiplication gives rise to a vector $\mathbf{w}^* = (w_1^*, w_4^*, w_{10}^*)^T$ of non-negative integers and does not alter the ordering among the components of $\mathbf{s}^* = A\mathbf{w}^*$. We thus obtain a graph $G^* = w_1^* G_1 + w_4^* G_4 + w_{10}^* G_{10}$ with s_2^*, s_3^* , and s_4^* in the prescribed order.

w_1	w_4	w_{10}	ordering of s_2, s_3, s_4
1	0	0	$s_2 > s_3 > s_4$
33	1	0	$s_3 > s_2 > s_4$
1	1	0	$s_3 > s_4 > s_2$
0	0	1	$s_4 > s_3 > s_2$
1999	0	1	$s_2 > s_4 > s_3$
1701	0	1	$s_4 > s_2 > s_3$

TABLE 1. We may select the parameters w_1, w_4, w_{10} in the graph G defined in (4) to obtain all six orderings of s_2, s_3, s_4 .

Table 1 displays the values of the parameters w_1, w_4 , and w_{10} that give rise to the six orderings of s_2, s_3 , and s_4 . (The values of r_1, r_4 , and r_{10} have been selected to yield convenient values of w_1, w_4 , and w_{10} .) Therefore the Roller-Coaster Conjecture holds for $\alpha = 4$.

Brown *et al* [3] conjectured that for a well-covered graph G with independence number α the roots of the independence polynomial $S(G, z)$ all fall in the disk $|z + \frac{\alpha}{2}| < \frac{\alpha}{2}$. The well-covered graph G corresponding to either of the last two rows of Table 1 serves as a counterexample to both this conjecture and the Unimodality Conjecture; two of the complex roots of $S(G, z) = 0$ lie outside the prescribed disk $|z + 2| < 2$.

We pass over $\alpha = 5$ for the moment and treat the case $\alpha = 6$. Consider the well-covered graph

$$G = G_1 + G_2 + G_6 + G_{10}, \quad \text{where } G_j = \underbrace{(6K_j + \cdots + 6K_j)}_{w_j} \quad (j = 1, 2, 4, 10).$$

A computation confirms that when $(w_1, w_2, w_7, w_{10}) = (18384800, 416745, 480, 343)$, G is a flat roller-coaster graph: the independence sequence (s_0, s_1, \dots, s_6) of G satisfies $s_3 = s_4 = s_5 = s_6$. As in the case $\alpha = 4$, by clearing denominators of suitable rational perturbations of w_1, w_2, w_7 , and w_{10} , we can construct a well-covered graph for each of the 24 orderings of s_3, s_4, s_5 , and s_6 .

4. EVEN TO ODD: THE CIRCULANT SUBSTITUTION

We treat the case $\alpha = 5$ of the Roller-Coaster Conjecture by modifying the construction of the flat roller-coaster graph in Section 3 for $\alpha = 4$. This modification uses a special circulant graph, which we now construct.

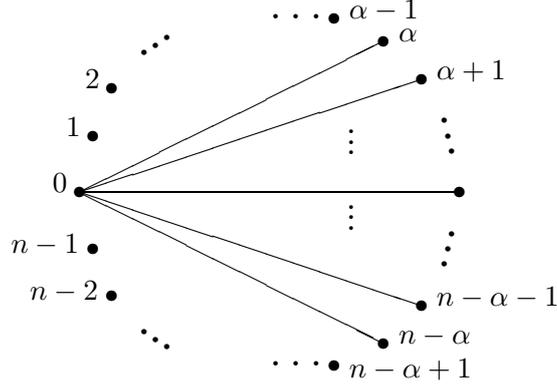


FIGURE 2. The circulant graph $C_n(\alpha)$. Only edges incident with vertex 0 are shown.

Let n and α be positive integers with $n \geq 3\alpha$. Let $C_n(\alpha) = (V, E)$ denote the circulant graph with vertex set $V = \{0, 1, \dots, n-1\}$, where vertex i is adjacent to vertices $i + \alpha, i + \alpha + 1, \dots, i + n - \alpha$, with vertex labels taken modulo n , as indicated in Figure 2. We claim that the independence polynomial of $C_n(\alpha)$ is

$$S(C_n(\alpha), z) = 1 + nz(1+z)^{\alpha-1}.$$

Vertex i is present in $\binom{\alpha-1}{k-1}$ independent k -subsets ($k \geq 1$) of the vertex subset $\{i, i+1, \dots, i+\alpha-1\}$, with vertex labels taken modulo n . As i runs through each of the n vertices, this scheme counts each independent k -set exactly once. Thus $s_k = n \binom{\alpha-1}{k-1}$ for $k \geq 1$, and the above formula follows.

Note that the independence sequence of the circulant graph $C_n(\alpha)$ is related to that of $(\alpha-1)K_1$ by an index shift and a rescaling:

$$S(C_n(\alpha), z) = 1 + nz(1+z)^{\alpha-1} = 1 + nz S((\alpha-1)K_1, z).$$

For $b \in \{1, 4, 10\}$ we let $G(b, n)$ be the lexicographic product of the circulant graph $C_n(5)$ and the complete graph K_b . (Thus each vertex in $C_n(5)$ is replaced by a complete graph K_b , and each edge is replaced by a complete bipartite graph $K_{b,b}$.) Each independent k -set of vertices in $C_n(5)$ gives rise to b^k independent k -sets in $G(b, n)$ for $k \geq 1$. Thus the independence polynomial of $G(b, n)$ is

$$S(G(b, n), z) = 1 + n(bz)(1+(bz))^4 \quad (b = 1, 4, 10).$$

Now we produce a flat roller-coaster graph for $\alpha = 5$ from a flat roller-coaster graph for $\alpha = 4$. Recall that with $(w_1, w_4, w_{10}) = (8000, 15, 4)$ in (4) we obtain a flat roller-coaster graph G for $\alpha = 4$. Now we replace each $4K_b$ in G with a join of copies of $G(b, n_b)$ to produce the graph

$$G^* = (4 \cdot 10 \cdot n_4 n_{10} w_1) G(1, n_1) + (1 \cdot 10 \cdot n_1 n_{10} w_4) G(4, n_4) + (1 \cdot 4 \cdot n_1 n_4 w_{10}) G(10, n_{10}).$$

The parameters have been chosen so that we obtain a common factor from the independence polynomials of each of the joined graphs in G^* . Thus

$$\begin{aligned} S(G^*, z) &= 1 \cdot 4 \cdot 10 (n_1 n_4 n_{10}) z [w_1(1+z)^4 + w_4(1+4z)^4 + w_{10}(1+10z)^{10}] + 1 \\ &= 40 (n_1 n_4 n_{10}) z S(G, z) + 1. \end{aligned}$$

Our construction shifts and rescales the terms of the independence sequence of the flat roller-coaster graph G to produce a graph with G^* whose independence sequence satisfies $s_3 = s_4 = s_5$. As in Section 3 we may now clear denominators of suitable rational perturbations of w_1, w_4 and w_{10} to obtain a well-covered graph for each of the six orderings of s_3, s_4 , and s_5 .

A similar modification of our construction for $\alpha = 6$ establishes the Roller-Coaster Conjecture for $\alpha = 7$. More generally, whenever we can satisfy the Roller-Coaster Conjecture for an *even* independence number α using joins of graphs of the form αK_b , then the circulant substitution can be applied directly to each ordering to establish the Roller-Coaster Conjecture for $\alpha + 1$.

5. PROBLEMS

Our attack on the Roller-Coaster Conjecture in Sections 3 and 4 prompts a general question.

Question 5. *Does there exist a well-covered graph with any given independence number α whose independence sequence satisfies $s_{\lceil \alpha/2 \rceil} = \cdots = s_\alpha$? In other words, do flat roller-coaster graphs exist for all α ?*

A counterexample to the Roller-Coaster Conjecture would lead to an intriguing problem.

Problem 6. *Suppose the Roller-Coaster Conjecture is false for well-covered graphs with independence number α . Characterize all permutations π of $\{\lceil \alpha/2 \rceil, \dots, \alpha\}$ for which there exists a well-covered graph whose independence sequence satisfies $s_{\pi(\lceil \alpha/2 \rceil)} < s_{\pi(\lceil \alpha/2 \rceil + 1)} < \cdots < s_{\pi(\alpha)}$.*

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