Differential Operators on Grassmann Varieties

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Abstract Following Weyl's account in *The Classical Groups* we develop an analogue of the (first and second) Fundamental Theorems of Invariant Theory for rings of differential operators: when V is a k-dimensional complex vector space with the standard $SL_k\mathbb{C}$ action, we give a presentation of the ring of invariant differential operators $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$ and a description of the ring of differential operators on the G.I.T. quotient, $D(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$, which is the ring of differential operators on the (affine cone over the) Grassmann variety of k-planes in n-dimensional space. We also compute the Hilbert series of the associated graded rings $GrD(\mathbb{C}[V^n])^{SL_k}$ and $Gr(D(\mathbb{C}[V^n]^{SL_k\mathbb{C}}))$. This computation shows that earlier claims that the kernel of the map from $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$ to $D(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$ is generated by the Casimir operator are incorrect. Something can be gleaned from these earlier incorrect computations though: the kernel meets the universal enveloping algebra of $\mathfrak{sl}_k\mathbb{C}$ precisely in the central elements of $U(\mathfrak{sl}_k\mathbb{C})$.

1 Introduction

If $R = \mathbb{C}[V] = \mathbb{C}[x_1,\dots,x_n]$ is a polynomial ring then the ring $D(R)$ of $\mathbb{C}$-linear differential operators [7, 12] on R, is just the Weyl algebra on V. That is, $D(R) = \mathbb{C}\langle x_1,\dots,x_n,\partial_1,\dots,\partial_n\rangle$, where $\partial_i = \partial/\partial x_i$ should be interpreted as the derivation on x_i and most of the variables commute, but we impose the relations

$$[\partial_i, x_i] = \partial_i x_i - x_i \partial_i = 1,$$

which just encode the usual product rule from Calculus. More generally, if X is an affine variety with coordinate ring $R = \mathbb{C}[X] = \mathbb{C}[V]/I$ then the ring of differential operators on X can be described as

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$$D(R) = D(\mathbb{C}[V]/I) = \frac{\{\theta \in D(\mathbb{C}[V]) : \theta(I) \subset I\}}{ID(\mathbb{C}[V])}. \tag{1}$$

In either setting, the ring of differential operators is a filtered algebra, with filtration given by the order of the operator. The associated graded ring $GrD(R)$ is a commutative ring and there is a map, the symbol map, from $D(R)$ to $GrD(R)$. We'll write ξ_i for the symbol of ∂_i.

When a group G acts on V, it not only induces an operator on the coordinate ring $\mathbb{C}[V]$, but also on the Weyl algebra $D(\mathbb{C}[V])$: if $\theta \in D(\mathbb{C}[V])$ and $g \in G$ then for all $f \in \mathbb{C}[V]$,

$$(g\theta)(f) = g(\theta(g^{-1}f)).$$

Our aim in the first part of this paper is to describe the ring of invariant differential operators $D(R)^G$ and the ring of differential operators $D(R^G)$ for certain classical rings of invariants R^G (with $R = \mathbb{C}[V^n]$ for a k-dimensional complex vector space V equipped with the usual $G = SL_k\mathbb{C}$ action). Though we might try to express R^G in the form $\mathbb{C}[V]/I$ and use (1), it is difficult to describe the set $\{\theta \in D(\mathbb{C}[V]) : \theta(I) \subset I\}$, so this is impractical. Rather, we exploit the natural map $\pi_* : D(R)^G \to D(R^G)$ given by restricting invariant operators to the invariant ring. The properties of this map are quite subtle – the reader is referred to Schwarz's detailed exposition [15] – but in general π_* need not be surjective and is almost never injective. Fortunately, Schwarz showed that in the cases that we are interested in, π_* is surjective with kernel equal to $D(R)\mathfrak{sl}_k\mathbb{C} \cap D(R)^G$. We exploit this to describe generators for $D(R^G)$. This is particularly interesting since $R^G = \mathbb{C}[V^n]^{SL_k\mathbb{C}}$ is the coordinate ring of the Grassmann variety $G(k,n)$ of k-planes in n-space (which is a cone over the projective Grassmann variety).

The Fundamental Theorem of Invariant Theory gives a presentation of the coordinate ring $\mathbb{C}[G(k,n)] = R^G$. Following a path strongly advocated by Weyl [19][1] we extend the Fundamental Theorem to $D(R)^G$, giving a presentation of the invariant differential operators on the affine variety $G(k,n)$. Because the action of G on R preserves the filtration, we also have graded rings $Gr(D(R))^G = Gr(D(R)^G)$ and $GrD(R^G)$. Applying the Fundamental Theorem to $GrD(R)^G$, we obtain generators and relations of the graded algebra. These lift to generators of $D(R)^G$ and each of the relations on the graded algebra extends to a relation on $D(R)^G$.

In conference talks based on two earlier papers [17, 18] I claimed that $ker(\pi_*)$ was generated by the Casimir operator. This is false. In the second part of this paper, we show that this cannot be true by computing the Hilbert series of $GrD(R)^G$, $GrD(R^G)$ and the graded image of $ker(\pi_*)$. However, the earlier computations were not entirely without merit: they predict that $ker(\pi_*) \cap U(\mathfrak{sl}_k\mathbb{C}) = Z(U(\mathfrak{sl}_k\mathbb{C}))$, a result that we prove using infinitesimal methods. The Hilbert series computations suggest that $GrD(R^G)$ may be Gorenstein; however, this remains an open question.

The invariant-theoretic methods of this paper are described in Derksen and Kemper's very nice book [3]. As previously mentioned, this paper also makes crucial use

[1] Also see the references in Olver [13], particularly those in Chapter 6.

of Schwarz's results in [15]. It is a pleasure to dedicate this paper to Gerry Schwarz on the occasion of his 60th birthday.

2 A Fundamental Theorem

Let V be a k-dimensional complex vector space and let V^* be the dual space of V. Then $\mathbb{C}[V^r \oplus (V^*)^s]$ is generated by the coordinates x_{ij} $(1 \le i \le k, 1 \le j \le r)$ and ξ_{ij} $(1 \le i \le k, 1 \le j \le s)$. Moreover, we have a natural $SL_k\mathbb{C}$ action on $\mathbb{C}[V^r \oplus (V^*)^s]$: $SL_k\mathbb{C}$ acts diagonally, on the V's by the standard representation and on the V^*'s by the contragredient representation. To be explicit, if $\mathbf{e}_i$ is the i^{th} standard basis vector then $A \in SL_k\mathbb{C}$ acts via:

$$\begin{aligned} A \cdot x_{ij} &= (x_{1j}, x_{2j}, \ldots, x_{kj}) A \mathbf{e}_i \\ \text{and} \quad A \cdot \xi_{ij} &= (\xi_{1j}, \xi_{2j}, \ldots, \xi_{kj}) (A^{-1})^T \mathbf{e}_i, \end{aligned}$$

where B^T is the usual transpose of B, $B^T_{ij} = B_{ji}$. To clarify with a simple example, if $r = s = 1$ and $k = 2$, then the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{C}$ acts on the variables $x_{11}, x_{21}, \xi_{11}, \xi_{21}$ to give

$$A \cdot x_{11} = ax_{11} + cx_{21},\ A \cdot x_{21} = bx_{11} + dx_{21},\ A \cdot \xi_{11} = d\xi_{11} - b\xi_{21},\ A \cdot \xi_{21} = -c\xi_{11} + a\xi_{21}.$$

If $\langle \cdot, \cdot \rangle : V \times V^* \to \mathbb{C}$ is the canonical pairing, for each $i \le r$ and $j \le s$ we have an invariant $\langle ij \rangle : V^r \oplus (V^*)^s \to \mathbb{C}$ that sends $(v_1, \ldots, v_r, w_1, \ldots, w_s)$ to $\langle v_i, w_j \rangle$. In coordinates $\langle ij \rangle = \sum_{\ell=1}^k x_{\ell i} \xi_{\ell j}$.

There are other invariants too. If $I = (I_1, I_2, \ldots, I_k)$ is a sequence with $1 \le I_1 < I_2 < \cdots < I_k \le r$, then we have a bracket invariant $[I] = [I_1 I_2 \cdots I_k] : V^r \oplus (V^*)^s \to \mathbb{C}$ given by

$$(v_1, \ldots, v_r, w_1, \ldots, w_s) \to \det(v_{I_1} v_{I_2} \cdots v_{I_k}).$$

This is a polynomial of degree k that only involves the x_{ij}. As well, if $J = (J_1, \ldots, J_k)$ is a sequence with $1 \le J_1 < J_2 < \cdots < J_k \le s$, then we have an invariant $|J| = |J_1 J_2 \cdots J_k| : V^r \oplus (V^*)^s \to \mathbb{C}$ given by

$$(v_1, \ldots, v_r, w_1, \ldots, w_s) \to \det(w_{J_1} w_{J_2} \cdots w_{J_k}).$$

This is a polynomial of degree k that only involves the ξ_{ij}.

The Fundamental Theorem of Invariant Theory describes the $SL_k\mathbb{C}$-invariants in the ring $\mathbb{C}[V^r \oplus (V^*)^s]$ and the relations among them (see [14, Sections 9.3 and 9.4]).

Theorem 1 (Fundamental Theorem of Invariant Theory). *Let V be a k-dimensional complex vector space. The invariant ring*

$$\mathbb{C}[V^r \oplus (V^*)^s]^{SL_k\mathbb{C}}$$

is generated by all $\langle ij \rangle$ *(*$1 \le i \le r, 1 \le j \le s$*), all* $[I] = [I_1 I_2 \cdots I_k]$ *(*$1 \le I_1 < I_2 < \cdots < I_k \le r$*) and all* $|J| = |J_1 J_2 \cdots J_k|$ *(*$1 \le J_1 < J_2 < \cdots < J_k \le s$*). The relations among these generators are of five types:*
(a) For $1 \le I_1 < I_2 < \cdots < I_k \le r$ *and* $1 \le J_1 < J_2 < \cdots < J_k \le s$*:*
$\det(\langle IJ \rangle) = \det(\langle I_a J_b \rangle)_{a,b=1}^{k} = [I_1 I_2 \cdots I_k]|J_1 J_2 \ldots J_k|$
(b) For $1 \le I_1 < I_2 < \cdots < I_{k+1} \le r$ *and* $1 \le j \le s$*:*
$\sum_{\ell=1}^{k+1} (-1)^{\ell-1} [I_1 I_2 \cdots \hat{I}_\ell \cdots I_{k+1}] \langle I_\ell j \rangle = 0$
(c) For $1 \le J_1 < J_2 < \cdots < J_{k+1} \le s$ *and* $1 \le i \le r$*:*
$\sum_{\ell=1}^{k+1} (-1)^{\ell-1} \langle i J_\ell \rangle |J_1 J_2 \cdots \hat{J}_\ell \cdots J_{k+1}| = 0$
(d) For $1 \le I_1 < I_2 < \cdots < I_{k-1} \le r$ *and* $1 \le J_1 < J_2 < \cdots < J_{k+1} \le r$*:*
$\sum_{\ell=1}^{k+1} (-1)^{\ell-1} [I_1 I_2 \cdots I_{k-1} J_\ell][J_1 J_2 \cdots \hat{J}_\ell \cdots J_{k+1}] = 0$
(e) For $1 \le I_1 < I_2 < \cdots < I_{k-1} \le s$ *and* $1 \le J_1 < J_2 < \cdots < J_{k+1} \le s$*:*
$\sum_{\ell=1}^{k+1} (-1)^{\ell-1} |I_1 I_2 \cdots I_{k-1} J_\ell| \, |J_1 J_2 \cdots \hat{J}_\ell \cdots J_{k+1}| = 0.$

Example 1. When $r = n$ and $s = 0$, Theorem 1 shows that $\mathbb{C}[V^n]^{SL_k\mathbb{C}}$ is generated by brackets $[I]$ satisfying the relations in (d). This invariant ring is the *bracket algebra*, the coordinate ring $\mathbb{C}[G(k,n)]$ of the Grassmann variety of k-planes in n-space. In this context, the generators are called the Plücker coordinates and the relations in part (d) are called the Grassmann-Plücker relations.

Now if V is a k-dimensional vector space, $(GrD(\mathbb{C}[V^n]))^{SL_k\mathbb{C}} = \mathbb{C}[V^n \oplus (V^*)^n]^{SL_k\mathbb{C}}$ so we can apply Theorem 1 in the case $r = s = n$ to compute $(GrD(\mathbb{C}[V^n]))^{SL_k\mathbb{C}}$. By [17, Theorem 1], $Gr(D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}) = (GrD(\mathbb{C}[V^n]))^{SL_k\mathbb{C}}$, so the lifts of the generators for $\mathbb{C}[V^n \oplus (V^*)^n]^{SL_k\mathbb{C}}$ generate $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$. Lifting the generators is easy since we only need to replace ξ_{ij} with ∂_{ij}.

Together with some very subtle work by Schwarz [15], this remark is also sufficient to determine the generators of the ring of differential operators $D(\mathbb{C}[V^n]^{SL_k\mathbb{C}}) = D(G(k,n))$ on the affine cone $\mathrm{Spec}(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$ over the Grassmann variety $G(k,n)$.

Theorem 2. *The ring of differential operators* $D(\mathbb{C}[G(k,n)])$ *on the affine cone over* $G(k,n)$ $(0 < k < n)$ *is generated by the images under* π_* *of the lifts to* $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$ *of the operators* $\langle ij \rangle$*,* $[I_1 I_2 \cdots I_k]$ *and* $|J_1 J_2 \cdots J_k|$*.*

Proof. Using Theorem 1 we see that these are the operators (with ξ_{ij} replacing ∂_{ij}) that generate $(GrD\mathbb{C}[V^n])^{SL_k\mathbb{C}} = \mathbb{C}[V^n \oplus (V^*)^n]^{SL_k\mathbb{C}}$. These lift to generators of $(D(\mathbb{C}[V^n]))^{SL_k\mathbb{C}}$. Though $SL_k\mathbb{C}$ does not in general satisfy the LS-alternative, this representation of $SL_k\mathbb{C}$ does satisfy the LS-alternative (see Schwarz [15, 11.6]) and so $\pi_* : (D(\mathbb{C}[V^n]))^{SL_k\mathbb{C}} \to D(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$ is surjective or $\mathbb{C}[V^n]^{SL_k\mathbb{C}}$ is smooth. Since the affine cone over $G(k,n)$ is singular for $0 < k < n$, π_* must be surjective. Thus these generators restrict to generators of $D(\mathbb{C}[V^n]^{SL_k\mathbb{C}}) = D(\mathbb{C}[G(k,n)])$. □

Now we turn to the relations among the generators of $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$. Each of the relations in Theorem 1 extends to an ordered relation on $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$. The relations in (b), (c), (d) and (e) apply unchanged in the noncommutative ring $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$. However, the relations in part (a) need to be modified: the meaning of the determinant in a noncommutative ring needs to be clarified[2] and there are

[2] If S is a noncommutative ring and $A \in M_{k \times k}(S)$ is a square matrix with entries in S then by the determinant of A we mean

lower order terms that need to be added to extend the relations to the non-graded ring. Fortunately, Capelli [1] found a beautiful way to incorporate the lower order terms by changing some of the terms in the determinant.[3] For each $k \in I \cap J$, the term $\langle kk \rangle$ in the matrix $\langle IJ \rangle$ should be replaced by

$$\langle kk \rangle + (\tau_I(k) - 1),$$

where $\tau_I(k)$ is the index of k in the sequence I – that is, $I_{\tau_I(k)} = k$.

Example 2. When $n = 4$ and $k = 3$, the extension of identity (a) for $[I] = [123]$ and $|J| = |124|$ is

$$\det \begin{bmatrix} \langle 11 \rangle + 0 & \langle 12 \rangle & \langle 14 \rangle \\ \langle 21 \rangle & \langle 22 \rangle + 1 & \langle 24 \rangle \\ \langle 31 \rangle & \langle 32 \rangle & \langle 34 \rangle \end{bmatrix} = [123]|124|.$$

Example 3. When $n = 4$ and $k = 3$, the extension of identity (a) for $[I] = [123]$ and $|J| = |234|$ is

$$\det \begin{bmatrix} \langle 12 \rangle & \langle 13 \rangle & \langle 14 \rangle \\ \langle 22 \rangle + 1 & \langle 23 \rangle & \langle 24 \rangle \\ \langle 32 \rangle & \langle 33 \rangle + 2 & \langle 34 \rangle \end{bmatrix} = [123]|234|.$$

To describe a presentation of the ring $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$ in terms of generators and relations we also need to incorporate the commutator relations. The ring $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$ is generated by the same operators that generate $\mathbb{C}[V^n \oplus (V^*)^n]^{SL_k\mathbb{C}}$. To present such a ring, take a noncommutative free algebra F in generators with the same names as the generators of $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$. The kernel K of the map $F \to D(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$ given by the natural identification of generators is a two-sided ideal of F.

Theorem 3. *The ideal K of relations is generated by the commutator relations and the extensions of each of the relations in Theorem 1.*

Proof. Let C be the two-sided ideal of F generated by the commutator relations among the generators of $D(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$. Then F/C is a filtered ring in which each of the generators $\langle ij \rangle$ ($[I]$, $|J|$,respectively) has order 1 (0, 2, respectively). The associated graded ring $Gr(F/C)$ is a commutative polynomial ring whose variables have the same names as those in F. Because the filtrations on F/C and $D(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$ are compatible, there is a surjective map $\delta : Gr(F/C) \to GrD(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$ and the kernel of δ is the image of the ideal K in the associated graded ring $Gr(F/C)$. However, the generators of $\ker(\delta)$ are given by the relations in Theorem 1. The lifts of these generators to F/C generate the kernel of the map $F/C \to D(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$ and

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$$\det(A) = \sum_{\sigma \in \mathbb{S}_k} \operatorname{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{k\sigma(k)}.$$

[3] I'm grateful to Minoru Itoh for pointing out Capelli's identity to me at a workshop held at Hokkaido University. In my original research I discovered a less elegant extension for the relations through experiments with the D-modules package of Macaulay2 [11, 4] and the Plural package of Singular [5, 6].

so these lifts, together with the commutator relations, generate the kernel of the map $F \to D(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$. $\square$

The generators and relations described so far are enough to give a presentation of $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$; that is, to determine the invariant differential operators. The map π_* from $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$ to $D(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$, the differential operators on the coordinate ring of the geometric quotient $\mathbb{C}[V^n]^{SL_k\mathbb{C}}$, is not injective. Its kernel is a two-sided ideal in $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$. Schwarz [15] showed that this kernel consists of the $SL_k\mathbb{C}$-stable part of the left ideal of $D(\mathbb{C}[V^n])$ generated by the operators in the Lie algebra $\mathfrak{sl}_k\mathbb{C}$. In several conference talks based on two previous papers [17, 18] I described an elimination computation in the case $k=2$, $n=4$ that seemed to show that this kernel is generated by a single element, the Casimir operator. However, I interpreted the results of this computation incorrectly: in fact $\ker(\pi_*)$ requires many generators. We'll show this by computing the Hilbert series for the graded image of $\ker(\pi_*)$ in the next section.

The elimination computation mentioned earlier actually shows something interesting that holds more generally: the center of the universal enveloping algebra of $\mathfrak{sl}_2\mathbb{C}$ is the part of the kernel of π_* that lies in $U(\mathfrak{sl}_2\mathbb{C})$.

Theorem 4. $\ker(\pi_*) \cap U(\mathfrak{sl}_k\mathbb{C}) = Z(U(\mathfrak{sl}_k\mathbb{C}))$.

Proof. Note that $\ker(\pi_*) \cap U(\mathfrak{sl}_k\mathbb{C}) = U(\mathfrak{sl}_k\mathbb{C})^{SL_k\mathbb{C}}$ since $\ker(\pi_*) = (D(\mathbb{C}[V^n])\mathfrak{sl}_k\mathbb{C})^{SL_k\mathbb{C}}$. An element of $U(\mathfrak{sl}_k\mathbb{C})$ is $SL_k\mathbb{C}$-stable if and only if it is annihilated by the adjoint action of the Lie algebra $\mathfrak{sl}_k\mathbb{C}$ (see, for example, Sturmfels [16, Lemma 4.5.1]). But if $\delta \in U(\mathfrak{sl}_k\mathbb{C})$ then

$$ad(\sigma)(\delta) = [\sigma, \delta] = 0 \text{ for all } \sigma \in \mathfrak{sl}_k\mathbb{C} \iff \delta \in Z(U(\mathfrak{sl}_k\mathbb{C})).$$

This shows that $U(\mathfrak{sl}_k\mathbb{C})^{SL_k\mathbb{C}} = Z(U(\mathfrak{sl}_k\mathbb{C}))$, as desired. $\square$

To close this section we emphasize that both $D(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$ and $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$ are noncommutative rings. This distinguishes these rings from the rings of operators considered in [8], which are commutative. For instance, when $k=2$ and $n=4$, let $\theta \bullet f$ represent the result of applying the operator θ to f and compute

$$(\langle 12\rangle[23]) \bullet [23] = (x_{11}\partial_{12} + x_{21}\partial_{22}) \bullet (x_{12}x_{23} - x_{22}x_{13})^2 = 2[23][13]$$

and $[23]\langle 12\rangle \bullet [23] = [23][13]$. It follows that the operators $[23]\langle 12\rangle$ and $\langle 12\rangle[23]$ are not equal in $D(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$ and $D(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$. So both rings are noncommutative.

3 The Hilbert Series of $GrD(R)^G$ and $GrD(R^G)$

We use the method described in sections 4.6.2, 4.6.3, and 4.6.4 of Derksen and Kemper's book [3] to compute the Hilbert series of $GrD(\mathbb{C}[V^4])^{SL_2\mathbb{C}}$. We sketch the method here, simplified to the $SL_2\mathbb{C}$ case, but the reader is referred to their book for details. Let T be a 1-dimensional torus acting on an n-dimensional vector space W. The set of characters $X(T)$ is a free rank 1 group; let z be a generator of $X(T)$, which we write in multiplicative notation. After a convenient choice of basis, the action of T on W is diagonal, given by the matrix

$$\rho = \begin{pmatrix} z^{m_1} & 0 & \cdots & 0 \\ 0 & z^{m_2} & 0 & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & z^{m_n} \end{pmatrix},$$

where the m_i are integers. The character of W is defined to be the trace of this representation, $\chi^W = z^{m_1} + z^{m_2} + \cdots + z^{m_n}$, and it follows that $\dim W^T$ is the coefficient of $z^0 = 1$ in χ^W. If $W = \oplus_{d=0}^{\infty} W_d$ is a graded vector space and W_d is a rational representation of T for each d, then define the T-Hilbert series of W to be

$$H_T(W,z,t) = \sum_{d=0}^{\infty} \chi^{W_d} t^d.$$

It follows that the Hilbert series $H(W^T,t) = \sum_{d=0}^{\infty} \dim(W_d^T) t^d$ is just the coefficient of $z^0 = 1$ in $H_T(W,z,t)$.

Now fix a maximal torus T in $SL_2\mathbb{C}$ and a Borel subgroup B of $SL_2\mathbb{C}$ containing T. Because the simple positive root of $SL_2\mathbb{C}$ is just twice the fundamental weight, we find that if $W = \oplus_{d=0}^{\infty} W_d$ is a graded vector space and W_d is a rational representation of T for each d, then the Hilbert series of the invariant space $H(W^G,t)$ is just the coefficient of $z^0 = 1$ in $(1-z^2)H_T(W,z,t)$; we are omitting a significant amount of detail here, the interested reader is referred to the argument on pages 186 and 187 of [3].

Example 4. Let V be a 2-dimensional complex vector space. If $W = GrD(\mathbb{C}[V^4])$ is made into a graded vector space using the *total degree order* then we can compute the Hilbert series of $GrD(\mathbb{C}[V^4])^{SL_2\mathbb{C}}$ as follows. First, note that $GrD(\mathbb{C}[V^4]) = \mathbb{C}[V^4 \oplus (V^*)^4]$ is a polynomial ring in 16 variables $x_{11}, \ldots, x_{24}, \xi_{11}, \ldots, \xi_{24}$ and that $SL_2\mathbb{C}$ contains a torus that acts on these variables with weight 1 (for x_{1i} and ξ_{2i}) or weight -1 (for x_{2i} and ξ_{1i})). Choosing the variables as a basis for the degree one part of $GrD(\mathbb{C}[V^4])$, we see that a maximal torus T acts diagonally on the degree one part of $GrD(\mathbb{C}[V^4])$ via a diagonal matrix with eight z's and eight z^{-1}'s along the diagonal. Since $W = GrD([V^4])$ is a polynomial ring generated by its degree one variables, we find that

$$H_T(W,z,t) = \sum_{d=0}^{\infty} \chi^{W_d} t^d = \frac{1}{(1-zt)^8(1-z^{-1}t)^8}.$$

Now we see that the Hilbert series of $GrD(\mathbb{C}[V^4])^{SL_2\mathbb{C}}$ is the coefficient of $z^0 = 1$ in the series expansion of $\frac{1-z^2}{(1-zt)^8(1-z^{-1}t)^8}$. To compute this, we note that the series converges if $|z^{-1}t| < 1$ and $|zt| < 1$. We will assume that $|z| = 1$ and $|t| < 1$. To find the coefficient of z^0, we divide by $2\pi iz$ and integrate over the unit circle C in $\mathbb{C}$ (in the positive orientation). So

$$\begin{aligned} H(GrD(\mathbb{C}[V^4])^{SL_2\mathbb{C}},t) &= \tfrac{1}{2\pi i}\int_C \tfrac{1-z^2}{z(1-zt)^8(1-z^{-1}t)^8}\,dz \\ &= \tfrac{1}{2\pi i}\int_C \tfrac{(1-z^2)z^7}{(1-zt)^8(z-t)^8}\,dz. \end{aligned}$$

This is the same as the sum of the residues inside C by the Residue Theorem. Since C has radius 1, and $|t|<1$, there is only one singularity of $\frac{(1-z^2)z^7}{(1-zt)^8(z-t)^8}$ inside C, namely at $z=t$. The residue there is the seventh coefficient in the Taylor series expansion of $g(z)=\frac{(1-z^2)z^7}{(1-zt)^8}$ about $z=t$. Computing $g^{(7)}(t)/7!$ gives

$$H(GrD(\mathbb{C}[V^4])^{SL_2\mathbb{C}},t) = \frac{1+15t^2+50t^4+50t^6+15t^8+t^{10}}{(1-t^2)^{13}}.$$

This has expansion $1+28t^2+\cdots$. The coefficient 28 refers to the 28 generators for $GrD(\mathbb{C}[V^4])^{SL_2\mathbb{C}}$: the sixteen $\langle ij\rangle$, the six brackets $[ij]$ and the six graded versions of the $|ij|$, $\xi_{1i}\xi_{2j}-\xi_{1j}\xi_{2i}$.

Now we compute the Hilbert series of the ideal $Gr(\ker(\pi_*))$, the image of $\ker(\pi_*)$ in the graded ring $GrD(\mathbb{C}[V^4])^{SL_2\mathbb{C}}$. This ideal is generated by the elements of $GrD(\mathbb{C}[V^4])\mathfrak{sl}_2\mathbb{C}$ that are invariant under the action of $SL_2\mathbb{C}$. The left ideal $GrD(\mathbb{C}[V^4])\mathfrak{sl}_2\mathbb{C}$ is generated by three operators

$$\begin{aligned} g_{12} &= x_{11}\xi_{21}+x_{12}\xi_{22}+x_{13}\xi_{23}+x_{14}\xi_{24}, \\ g_{21} &= x_{21}\xi_{11}+x_{22}\xi_{12}+x_{23}\xi_{13}+x_{24}\xi_{14}, \\ g_{11}-g_{22} &= x_{11}\xi_{11}+x_{12}\xi_{12}+x_{13}\xi_{13}+x_{14}\xi_{14}-x_{21}\xi_{21}-x_{22}\xi_{22}-x_{32}\xi_{32}-x_{42}\xi_{42}. \end{aligned}$$

These are all eigenvectors under the torus action and the torus acts with weights $2,-2$ and 0 on g_{12}, g_{21} and $g_{11}-g_{22}$, respectively. These three polynomials form a regular sequence in $GrD(\mathbb{C}[V^4])$ and we have an $SL_2\mathbb{C}$-equivariant resolution

$$\begin{aligned} 0 \to GrD(\mathbb{C}[V^4])(-6) &\to GrD(\mathbb{C}[V^4])(-4)^3 \\ \to GrD(\mathbb{C}[V^4])(-2)^3 &\to GrD(\mathbb{C}[V^4])\mathfrak{sl}_2\mathbb{C} \quad \to 0. \end{aligned}$$

The rightmost map sends the generators of $GrD(\mathbb{C}[V^4])(-2)^3$ to the three generators of $GrD(\mathbb{C}[V^4])\mathfrak{sl}_2\mathbb{C}$ and so the three generators of $GrD(\mathbb{C}[V^4])(-2)^3$ are equipped with torus weights -2, 2 and 0. Similarly, the three generators of $GrD(\mathbb{C}[V^4])(-4)^3$ have torus weights -2, 2 and 0, while the generator of the leftmost module has torus weight 0. The T-Hilbert series of these modules are:

$$\begin{aligned} H_T(GrD(\mathbb{C}[V^4])(-6),z,t) &= \tfrac{t^6}{(1-zt)^8(1-z^{-1}t)^8} \\ H_T(GrD(\mathbb{C}[V^4])(-4)^3,z,t) &= \tfrac{t^4(z^2+1+z^{-2})}{(1-zt)^8(1-z^{-1}t)^8} \\ H_T(GrD(\mathbb{C}[V^4])(-2)^3,z,t) &= \tfrac{t^2(z^2+1+z^{-2})}{(1-zt)^8(1-z^{-1}t)^8}. \end{aligned}$$

As in Example 4, to find the Hilbert series of the $SL_2\mathbb{C}$-invariants, we multiply by $(1-z^2)$ and find the $z^0 = 1$ coefficient in the resulting expression. This produces the Hilbert series:

$$\begin{aligned}
H(GrD(\mathbb{C}[V^4])(-6)^{SL_2\mathbb{C}},t) &= \tfrac{t^6+15t^8+50t^{10}+50t^{12}+15t^{14}+t^{16}}{(1-t^2)^{13}} \\
H([GrD(\mathbb{C}[V^4])(-4)^3]^{SL_2\mathbb{C}},t) &= \tfrac{36t^6+162t^8+162t^{10}+36t^{12}}{(1-t^2)^{13}} \\
H([GrD(\mathbb{C}[V^4])(-2)^3]^{SL_2\mathbb{C}},t) &= \tfrac{36t^4+162t^6+162t^8+36t^{10}}{(1-t^2)^{13}}.
\end{aligned}$$

Since the resolution was $SL_2\mathbb{C}$-equivariant, we get a resolution of the invariant modules:

$$\begin{aligned}
0 \to GrD(\mathbb{C}[V^4])(-6)^{SL_2\mathbb{C}} &\to \left(GrD(\mathbb{C}[V^4])(-4)^3\right)^{SL_2\mathbb{C}} \\
\to \left(GrD(\mathbb{C}[V^4])(-2)^3\right)^{SL_2\mathbb{C}} &\to \left(GrD(\mathbb{C}[V^4])\mathfrak{sl}_2\mathbb{C}\right)^{SL_2\mathbb{C}} \quad \to 0.
\end{aligned}$$

Since the alternating sum of the Hilbert series over an exact sequence is zero, we can use this data to determine the Hilbert series for $Gr\ker(\pi_*) = \left(GrD(\mathbb{C}[V^4])\mathfrak{sl}_2\mathbb{C}\right)^{SL_2\mathbb{C}}$,

$$\begin{aligned}
H(Gr\ker(\pi_*),t) &= \tfrac{36t^4+162t^6+162t^8+36t^{10}}{(1-t^2)^{13}} - \tfrac{36t^6+162t^8+162t^{10}+36t^{12}}{(1-t^2)^{13}} \\
&\quad + \tfrac{t^6+15t^8+50t^{10}+50t^{12}+15t^{14}+t^{16}}{(1-t^2)^{13}} \\
&= \tfrac{36t^4+127t^6+15t^8-76t^{10}+14t^{12}+15t^{14}+t^{16}}{(1-t^2)^{13}}.
\end{aligned}$$

In two previous papers [17, 18] I claimed that $\ker(\pi_*)$ was generated by the Casimir operator, an operator of total degree 4. However, the degree 4 part of this ideal is a 36-dimensional vector space (rather than a 1-dimensional space), so this claim is false. In fact $\ker(\pi_*)$ requires many generators.

Finally, by taking the difference of the Hilbert series for $GrD(\mathbb{C}[V^4])^{SL_2\mathbb{C}}$ and $Gr\ker(\pi_*)$, we find the Hilbert series for $GrD(\mathbb{C}[V^4]^{SL_2\mathbb{C}}) = GrD(G(2,4))$ in the total degree order,

$$H(GrD(G(2,4)),t) = \frac{1+18t^2+65t^4+65t^6+18t^8+t^{10}}{(1-t^2)^{10}}.$$

The form of this Hilbert series prompts us to ask whether $GrD(\mathbb{C}[V^4]^{SL_2\mathbb{C}})$ is Gorenstein. By a result due to R. Stanley (see [2, Corollary 4.4.6]) it is enough to show that $GrD(\mathbb{C}[V^4]^{SL_2\mathbb{C}})$ is Cohen-Macaulay. By the Hochster-Roberts Theorem [9] (or see [10, Theorem 3.6]), $GrD(\mathbb{C}[V^n])^{SL_k\mathbb{C}}$ is always Cohen-Macaulay, but I see no particular reason for the same to be true of $GrD(\mathbb{C}[V^n]^{SL_k\mathbb{C}})$.

Open Question: When are the graded rings $GrD(\mathbb{C}[V^n]^{SL_k\mathbb{C}}) = GrD(G(k,n))$ Gorenstein?

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