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Differential operators on orbifolds

William N. Traves*

Department of Mathematics, US Naval Academy, Mail Stop 9E, Annapolis, MD 21402, USA

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Abstract

An algorithm is presented that computes explicit generators for the ring of differential operators on an orbifold, the quotient of a complex vector space by a finite group action. The algorithm also describes the relations among these generators. The algorithm presented in this paper is based on Schwarz's study of a map carrying invariant operators to operators on the orbifold and on an algorithm to compute rings of invariants using Gröbner bases due to Derksen [Derksen, Harm, 1999. Computation of invariants for reductive groups. Adv. Math. 141 (2), 366–384]. It is also possible to avoid using Derksen's algorithm, instead relying on the Reynolds operator and the Molien series.
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Following the discouraging finding that rings of differential operators on singular varieties can be non-noetherian (Bernšteĭn et al., 1972), there has been much progress on identifying good properties enjoyed by rings of differential operators on special classes of singular varieties. In particular, much attention has been paid to quotient varieties; see, for example, Schwarz (1995), Levasseur and Stafford (1989) for classical group actions, and Musson and Van den Bergh (1998) for quotients by tori. When the group is finite, the ring of differential operators on the quotient space is a finitely generated algebra. The finite group case was treated by Kantor (1977), Ishibashi (1985), Levasseur (1981), Levasseur and Stafford (1995), Schwarz (1995) and more recently by Plesken and Robertz (2005). Those papers contain many ad hoc examples of rings of differential operators on orbifolds, the quotient of a complex vector space by a finite group. In this paper we give an algorithm that not only computes explicit generators for the ring of differential operators on an orbifold but also describes the relations among these generators.

* Tel.: +1 410 293 6774; fax: +1 410 293 4883.
E-mail address: traves@usna.edu.

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The methods presented in this paper were inspired by Schwarz's study of the map π_* carrying invariant operators to operators on the orbifold. One step of the algorithm involves computing a ring of invariants of a finite group acting on a commutative ring. To this end we describe Derksen's algorithm (Derksen, 1999), which uses Gröbner bases to compute rings of invariants. Both Derksen (1999) and Schwarz (1995) deal with more general algebraic groups, but for the purposes of this paper, we restrict our attention to the application of their methods for finite groups. All the subtleties of the algorithm already appear in this case.

Basic definitions, together with a description of Derksen's algorithm, are reviewed in Section 1, where the algorithm to compute rings of differential operators is stated explicitly. We give two examples to illustrate the algorithm in action in Section 2. In the last section we show that it is possible to replace Derksen's algorithm with a computation involving the Molien series and the Reynolds operator, though in general this gives a redundant system of generators.

1. Essential ingredients for the algorithm

Let G be a finite group that acts on an n-dimensional complex vector space V and let $R := \mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_n]$ be the coordinate ring of V, where the x_i are the coordinate functions. Collapsing each G-orbit to a point leads to an orbifold, the quotient variety V/G. The coordinate ring of V/G can be described as a ring of invariants: the action of G on V induces a natural action of G on $\mathbb{C}[V]$ via $g \cdot f(x) = f(g^{-1} \cdot x)$ and the coordinate ring $\mathbb{C}[V/G]$ is the subring R^G of R consisting of the invariant polynomials, $R^G = \{f \in R : \ g \cdot f = f \text{ for all } g \in G\}$. Our goal in this paper is to describe the ring of differential operators $D(R^G)$.

Rings of differential operators: Perhaps the most important ring of differential operators is the Weyl algebra,

$$D(\mathbb{C}[x_1, \ldots, x_n]) = \mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle,$$

where the variables x_i commute with each other, the variables $\partial_j = \partial/\partial x_j$ commute with each other, and the two sets of variables interact via the product rule: $\partial_j x_i = x_i \partial_j + \delta_{ij}$. If $S = \mathbb{C}[x_1, \ldots, x_n]/I$ is the coordinate ring of an affine algebraic variety then the ring of differential operators $D(S)$ can be described in terms of the Weyl algebra of the ambient space:

$$D(S) = \frac{\{\theta \in D(\mathbb{C}[x_1, \ldots, x_n]) : \ \theta \cdot I \subset I\}}{I D(\mathbb{C}[x_1, \ldots, x_n])}.$$

So one way to describe $D(R^G)$ is to present R^G as a quotient of a polynomial ring. Unfortunately, the description of the set $\{\theta \in D(\mathbb{C}[x_1, \ldots, x_n]) : \ \theta \cdot I \subset I\}$ may require prohibitive syzygy computations.

We take a different approach, based on realizing R^G as a subring of R. It would be best to describe $D(R^G)$ as generated by the restriction of elements of the Weyl algebra $D(R)$ to R^G. However, this is not always possible (see Example 4). Instead we find a polynomial subring $\mathbb{C}[z_1, \ldots, z_n] \subset R$ and describe the generators of $D(R^G)$ as the restriction of operators in the Weyl algebra $\mathbb{C}\langle z_1, \ldots, z_n, \partial_1, \ldots, \partial_n\rangle$ to R^G.

To describe $D(R^G)$ in these terms, we need a more general definition of the ring of differential operators, based on iterated commutators and due to Grothendieck (1967). If S is a commutative $\mathbb{C}$-algebra, then an endomorphism $\theta \in \mathrm{End}_{\mathbb{C}}(S)$ is said to have order $\leq n$ if for all $s_0, \ldots, s_n \in S$,

$$[\cdots[[\theta, s_0], s_1], \ldots, s_n] = 0,$$

where $[\theta, s] = \theta s - s\theta$ is the commutator of θ and s in $\mathrm{End}_{\mathbb{C}}(S)$. The ring of differential operators $D(S)$ consists of all endomorphisms of S with finite order. This is an unwieldy definition, but we will be able to avoid explicit computations with commutators in what follows.

Filtrations and gradings: The ring $D(S)$ is filtered by order: if $\mathcal{F}_k$ is the set of operators of order $\leq k$, then $\mathcal{F}_k\mathcal{F}_\ell \subset \mathcal{F}_{k+\ell}$. We form the graded ring $GrD(S) = \oplus_k \mathcal{F}_k/\mathcal{F}_{k-1}$ and define the symbol map $\sigma : D(S) \to GrD(S)$ so that $\sigma(\theta)$ is the image of θ in $\mathcal{F}_k/\mathcal{F}_{k-1}$ if $\theta \in \mathcal{F}_k$ and $\theta \notin \mathcal{F}_{k-1}$. For brevity we will often write $\bar{\theta}$ for $\sigma(\theta)$. Now if θ has order $\leq k$ and γ has order $\leq \ell$ it is easy to check by induction that $\theta\gamma - \gamma\theta$ has order $\leq k + \ell - 1$, so the graded ring $GrD(S)$ is a commutative ring. In particular, the ring $GrD(R)$ is just a polynomial ring in $2n$ variables: $GrD(R) = \mathbb{C}[x_1, \ldots, x_n, \bar{\partial}_1, \ldots, \bar{\partial}_n]$.

The G-action on $R = \mathbb{C}[x_1, \ldots, x_n]$ induces a G-action on both $D(R)$ and $GrD(R)$. If $\theta \in D(R)$ then $(g \cdot \theta)(r) = g \cdot (\theta(g^{-1} \cdot r))$ and both $g \cdot \theta$ and θ have the same order. We define the G-action on $GrD(R)$ to be compatible with the symbol map: $g \cdot \sigma(\theta) := \sigma(g \cdot \theta)$. This is a well-defined action since if $\sigma(\theta) = \sigma(\gamma)$ then $\mathrm{order}(g \cdot (\theta - \gamma)) = \mathrm{order}(\theta - \gamma) < \mathrm{order}(\theta) = \mathrm{order}(\gamma)$ and so $\sigma(g \cdot \gamma) = \sigma(g \cdot \gamma + g \cdot (\theta - \gamma)) = \sigma(g \cdot \theta)$.

It is helpful to write out the action of G on $D(R)$ and $GrD(R)$ explicitly. If $g \in G$ acts on $R = \mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}[V]$ via the matrix A, $g \cdot \mathbf{x} = A\mathbf{x}$ (all vectors are represented by column matrices), then g acts on $\mathbb{C}[V^*] = \mathbb{C}[\partial] = \mathbb{C}[\partial_1, \ldots, \partial_n]$ via $(A^{\mathrm{T}})^{-1}$, where A^{T} is the Hermitian transpose (A^{T}_{ij} is the complex conjugate of A_{ji}). To see this,[1] think of $D(R)$ as a subset of $\mathrm{Hom}_{\mathbb{C}}(R, R)$. If $\partial \in V^*$ and $x \in V$ then

$$(g \cdot \partial)(x) = g \cdot \partial(x \circ g^{-1}) = \partial(x \circ g^{-1}) = ((g^{-1})^{\mathrm{T}}\partial)(x).$$

Of course, G acts on $GrD(R)$ in a manner compatible with the symbol map. So $g \in G$ also acts on the $\sigma(\partial_i)$'s via the matrix $(A^{\mathrm{T}})^{-1}$. It is important to note that the action of G operates separately on the variables x_i and the derivations ∂_j and does not mix the two sets of variables.

Theorem 1. *$Gr(D(R)^G)$ is canonically isomorphic to $(GrD(R))^G$.*

Proof. If $\theta \in D(R)^G$ then $\bar{\theta} \in (GrD(R))^G$: for all $g \in G$, $g \cdot \sigma(\theta) = \sigma(g \cdot \theta) = \sigma(\theta)$. This shows that $Gr(D(R)^G)$ can be canonically identified with a subset of $(GrD(R))^G$. As well, if $\delta \in (GrD(R))^G$, then there is an operator $\theta \in D(R)$ with $\bar{\theta} = \delta$. Indeed, writing $\delta = \sum_{\mathbf{a},\mathbf{b}} m_{\mathbf{a},\mathbf{b}} x^{\mathbf{a}} \bar{\partial}^{\mathbf{b}}$, we can take $\theta = \sum_{\mathbf{a},\mathbf{b}} m_{\mathbf{a},\mathbf{b}} x^{\mathbf{a}} \partial^{\mathbf{b}} \in D(R)$. Since the action of G on x and ∂ is the same as the action on x and $\bar{\partial}$, $\delta \in (GrD(R))^G$ forces $\theta \in D(R)^G$. This shows that the subset in question is the entire ring $(GrD(R))^G$. $\square$

The graded ring $Gr(D(R)^G)$ is a powerful tool to study $D(R)^G$. In particular, if $\bar{\theta}_1, \ldots, \bar{\theta}_k$ generate the commutative ring $Gr(D(R)^G)$ then $\theta_1, \ldots, \theta_k$ generate $D(R)^G$.

The symbol map suggests that the natural grading on $GrD(R)$ comes from the order filtration: we assign the variables x_i degree zero and the $\bar{\partial}_j$ degree 1. Like all polynomial rings, $GrD(R)$ has many graded structures. We will find it convenient to sometimes work with the total degree, assigning degree 1 to both the x_i and the $\bar{\partial}_i$. This has the advantage that the graded pieces of $GrD(R)$ are finite-dimensional vector spaces.

[1] I am grateful to an anonymous referee for the following simple explanation.

Relating $D(R)^G$ to $D(R^G)$: The projection $\pi : V \to V/G$ corresponds to the embedding $R^G \hookrightarrow R$ and this induces a map $\pi_* : D(R)^G \to D(R^G)$ via restriction. To make sense out of this statement, we check that if $\theta \in D(R)^G$ and $r \in R^G$ then $(\pi_*\theta)(r) := \theta(r)$ is in $D(R^G)$:

$$g \cdot (\theta(r)) = g \cdot (\theta(g^{-1} \cdot r)) = (g \cdot \theta)(r) = \theta(r).$$

We will investigate the behavior of the map π_* in the case that G is a finite group (more general situations are covered by Kantor (1977), Levasseur (1981) and Schwarz (1995)).

Theorem 2. *The map π_* is injective when G is a finite group.*

Proof. This is Theorem 6.3(1) in Schwarz (1995) but there the proof relies on more general results using vector bundles. We give an elementary proof. We need to show that if $\theta \in D(R)$ and $\theta(R^G) = 0$ then $\theta = 0$. Clearly this is true when order$(\theta) = 0$ since then θ is a multiplication operator on the domain R. We suppose that θ is a counterexample of minimal order (≥ 1) and aim for a contradiction. If $x, y \in R^G$, then $[\theta, x](y) = \theta(xy) - x\theta(y) = 0$, so $[\theta, x](R^G) = 0$. Since $[\theta, x]$ has lower order than θ, $[\theta, x] = 0$ and θ is R^G-linear. If $h \in R$ and $x \in R^G$ then

$$[[\theta, h] - \theta(h)](x) = \theta(hx) - h\theta(x) - \theta(h)x = -h\theta(x) = 0.$$

Since $[\theta, h] - \theta(h)$ is an operator of lower order than θ that vanishes on R^G, $[\theta, h] = \theta(h)$. Thus $\theta(h^k) = h\theta(h^{k-1}) + h^{k-1}\theta(h) = \cdots = kh^{k-1}\theta(h)$. Since G is finite, R is integral over R^G (h satisfies the monic polynomial $\prod_{g \in G}(X - g \cdot h)$ whose coefficients are clearly in R^G). Let

$$Q(h) = h^d + a_1 h^{d-1} + \cdots + a_{d-1}h + a_d = 0 \quad (a_i \in R^G)$$

be the monic relation of minimal degree. Applying θ yields

$$(dh^{d-1} + a_1(d-1)h^{d-2} + \cdots + a_{d-1})\theta(h) = 0.$$

Since $Q(h)$ has minimal degree, $(dh^{d-1} + a_1(d-1)h^{d-2} + \cdots + a_{d-1}) \neq 0$. Thus $\theta(h) = 0$ since R is a domain. So $\theta = 0$ and we get our contradiction. □

Whether or not π_* is surjective depends on whether G contains any elements acting as pseudoreflections on V. A pseudoreflection is an element $g \in G$ whose fixed set V^g has codimension 1 (since G acts linearly on V, this forces V^g to be a hyperplane). Kantor (1977, Theorem 4 in 3.3.1) characterized when π_* is surjective.

Theorem 3. *The map $\pi_* : D(R)^G \to D(R^G)$ is surjective if and only if G contains no pseudoreflections.*

Example 4. Let $G = \mathbb{Z}_2 = \langle \gamma \rangle$ act on $R = \mathbb{C}[x]$ via $\gamma(x) = -x$. Then G is a reflection group and $R^G = \mathbb{C}[x^2]$. Writing $z = x^2$, $D(R^G)$ is the Weyl algebra $\mathbb{C}\langle z, \partial/\partial z \rangle$. Unfortunately, $\partial/\partial z$ does not extend to an operator on $\mathbb{C}[x]$ (see Schwarz (1995, Example 5.7)).

We will make use of Theorem 3 to obtain generators of $D(R^G)$ from generators of $D(R)^G$. The theorem deals with the case when no element of G is a pseudoreflection. The opposite case, when G is generated by pseudoreflections, is also interesting. If G is generated by pseudoreflections, then G is said to be a reflection group.

Theorem 5 (*Sheppard–Todd–Chevalley*). *The ring R^G is a polynomial ring (and $D(R^G)$ is a Weyl algebra) if and only if G is a reflection group.*

When the group G contains some pseudoreflections but is not a reflection group, then we factor G as follows. Let W be the subgroup generated by pseudoreflections in G. Note that W itself is a reflection group and that W is a normal subgroup of G. To see this it is enough to check that if w is a pseudoreflection and $g \in G$, then $gwg^{-1} \in W$; this follows since $(g \cdot V^w) \subset V^{gwg^{-1}}$ so $\operatorname{codim}(V^{gwg^{-1}}) \leq \operatorname{codim}(g \cdot V^w) = 1$. Now G/W acts on the polynomial ring R^W and $R^G = (R^W)^{G/W}$. Since G/W contains no pseudoreflections, the map $\pi_* : D(R^W)^{G/W} \to D(R^G)$ is an isomorphism.

Computing rings of invariants: There are many good algorithms to compute invariant rings of finite group actions. Kemper develops algorithms and describes their implementation in a beautiful survey paper (Kemper, 1998). For concreteness, we will describe another algorithm, due to Derksen (1999), that uses Gröbner bases to compute rings of invariants such as R^W and $(GrD(R))^G$. Derksen's algorithm applies to all linearly reductive groups (see Derksen and Kemper's excellent monograph (Derksen and Kemper, 2002)), but for simplicity we just describe it for finite groups acting on vector spaces. First we represent the group G and its action on $R = \mathbb{C}[x_1, \ldots, x_n]$ in terms of coordinates. As a set the group G can be identified with a finite set of points on an affine variety: G is identified with the vanishing set of some polynomials $f_1, \ldots, f_t$ in a polynomial ring $\mathbb{C}[t_1, \ldots, t_d]$ (it suffices to use one parameter t_1 and one relation $f_1(t_1) = t_1^{|G|} - 1$, but in some situations it is easier to parameterize the group by more than one parameter). The action of G on $R = \mathbb{C}[x_1, \ldots, x_n]$ is encoded as a map $\rho : G \to \operatorname{Aut}_{\mathbb{C}}(R)$. If $\mathbf{t} = (t_1, \ldots, t_d)$ represents an element of G – so that $f_1(\mathbf{t}) = \cdots = f_t(\mathbf{t}) = 0$ – then define elements $A_{ij}(\mathbf{t})$ such that $\rho(\mathbf{t})(x_i) = A_{i1}(\mathbf{t})x_1 + \cdots + A_{in}(\mathbf{t})x_n$. With this notation, $\rho(\mathbf{t})$ is given by the matrix $(A_{ij}(\mathbf{t}))$.

Example 6. If $G = \mathbb{Z}_2 = \{e, \gamma\}$, then represent G as the zero set of $f_1(t) = t^2 - 1$ with, for example, the root $t = 1$ representing the identity e and $t = -1$ representing γ. Suppose that $\gamma \in G$ acts on the ring $R = \mathbb{C}[x_1, x_2, x_3]$ by $\gamma(x_1) = -x_1$, $\gamma(x_2) = x_3$ and $\gamma(x_3) = x_2$. Then

$$\rho(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \rho(-1) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

This can be achieved by interpolation: $\rho(t) = \begin{bmatrix} t & 0 & 0 \\ 0 & \frac{t+1}{2} & \frac{1-t}{2} \\ 0 & \frac{1-t}{2} & \frac{t+1}{2} \end{bmatrix}$.

Derksen's algorithm relies on the Reynolds operator $\mathcal{R}$, an R^G-linear map from R to R^G that acts as the identity on R^G. When G is a finite group, the Reynolds operator is just an averaging map: $\mathcal{R}(f)(x) = \frac{1}{|G|}\sum_{g \in G}(g \cdot f)(x) = \frac{1}{|G|}\sum_{g \in G} f(g^{-1} \cdot x)$.

The key idea to find algebra generators for R^G goes back to Hilbert, who related these generators to the generators of an ideal in R. The Hilbert ideal I is the ideal of R that is generated by all homogeneous invariants of positive degree. Suppose that we have found a generating set for I, consisting of (not necessarily invariant) homogeneous polynomials $h_1, \ldots, h_k$ in R. Then an easy argument shows that $\mathcal{R}(h_1), \ldots, \mathcal{R}(h_k)$ generate R^G as an algebra (Derksen and Kemper, 2002, Theorem 2.2.10).

To find generators for the Hilbert ideal I, we take an indirect approach. Let $V = \operatorname{Spec}(R)$ and define the map of varieties

$$\begin{aligned}\Psi : G \times V &\longrightarrow V \times V \\ (g, v) &\longmapsto (v, g \cdot v).\end{aligned}$$

Let Y denote the image of Ψ and let $\mathbb{I}(Y)$ be the ideal defining the closed scheme Y. We represent G as $\mathrm{Spec}(\mathbb{C}[t_1, \ldots, t_d]/(f_1, \ldots, f_t))$ and the V in $G \times V$ and the first V in $V \times V$ as $\mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_n])$. The second V factor in $V \times V$ is represented using a different set of variables: $\mathrm{Spec}(\mathbb{C}[y_1, \ldots, y_n])$. The following relationship between $\mathbb{I}(Y)$ and the Hilbert ideal I was proved by Derksen using a subtle application of the Reynolds operator (see Derksen (1999) or Derksen and Kemper (2002, Theorem 4.1.3)).

Theorem 7. $\mathbb{I}(Y) + (y_1, \ldots, y_n) = I + (y_1, \ldots, y_n)$.

The ideal $\mathbb{I}(Y)$ can be computed by eliminating variables from the graph of Ψ. The graph of Ψ is the subset $\{(g, v, v, g \cdot v) : g \in G, v \in V\} \subset (G \times V) \times (V \times V)$, but since the two inner coordinates are equal, it is equivalent to just consider $\Gamma = \{(g, v, g \cdot v) : g \in G, v \in V\} \subset G \times V \times V$. This is a variety in $G \times V \times V$ defined by the ideal

$$\mathbb{I}(\Gamma) = (f_1(\mathbf{t}), \ldots, f_t(\mathbf{t})) + (\{y_i - (A_{i1}(\mathbf{t})x_1 + \cdots + A_{in}(\mathbf{t})x_n)\}_{i=1,\ldots,n})$$

in the ring $\mathbb{C}[G \times V \times V] = \mathbb{C}[t_1, \ldots, t_d, x_1, \ldots, x_n, y_1, \ldots, y_n]/(f_1(\mathbf{t}), \ldots, f_t(\mathbf{t}))$. Now $\mathbb{I}(Y)$ can be expressed as $\mathbb{I}(\Gamma) \cap \mathbb{C}[V \times V] = \mathbb{I}(\Gamma) \cap \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. This can be computed using a Gröbner basis (Kreuzer and Robbiano, 2000). Select an elimination order $>$ such that the t_i's are greater than any x_i's or y_i's (for example, lex order with $t_1 > t_2 > \cdots > t_d > x_1 > \cdots > x_n > y_1 > \cdots > y_n$). Compute a Gröbner basis B for $\mathbb{I}(\Gamma)$ using the order $>$. Then the elements of this basis that do not involve any t_i's are a generating set for $\mathbb{I}(Y)$.

Having obtained generators $h_1(x, y), \ldots, h_k(x, y)$ for $\mathbb{I}(Y)$, we can use Theorem 7 to obtain generators for I by just setting the y_i's to zero. Finally, applying the Reynolds operator to the $h_i(x, 0)$ gives a generating set for R^G.

Example 8. We compute R^G where $G = \mathbb{Z}_2$, $R = \mathbb{C}[x_1, x_2, x_3]$ and the action is as given in Example 6. Following the algorithm described above, we need to compute the Gröbner basis $\mathcal{B}$ of the ideal $\mathbb{I}(\Gamma) = (t^2 - 1, y_1 - tx_1, y_2 - \frac{t+1}{2}x_2 - \frac{1-t}{2}x_3, y_3 - \frac{1-t}{2}x_2 - \frac{t+1}{2}x_3)$ using a lex order that places the variable t before the others. Considering only $\mathcal{B} \cap \mathbb{C}[x_1, x_2, x_3, y_1, y_2, y_3]$ and setting $y_1 = y_2 = y_3 = 0$ we obtain generators for I: $I = (x_2 + x_3, x_3^2, x_1x_3, x_1^2)$. Applying the Reynolds operator – $\mathcal{R}(f(x_1, x_2, x_3)) = [f(x_1, x_2, x_3) + f(-x_1, x_3, x_2)]/2$ – to these generators gives algebra generators for R^G: R^G is the subalgebra of R generated by $x_2 + x_3, x_2^2 + x_3^2, -x_1x_2 + x_1x_3$, and x_1^2. To present R^G as a ring, we compute the ideal of relations

$$\begin{pmatrix} a - (x_2 + x_3), & b - (x_2^2 + x_3^2), \\ c - (-x_1x_2 + x_1x_3), & d - x_1^2 \end{pmatrix} \mathbb{C}[x_1, x_2, x_3, a, b, c, d] \cap \mathbb{C}[a, b, c, d]$$

using an elimination order that eliminates the first three variables. This leads to the presentation

$$R^G = \frac{\mathbb{C}[a, b, c, d]}{(a^2d + c^2 - 2bd)}$$

and R^G is isomorphic to a degree 4 hypersurface in a weighted projective space (here a has degree 1 and b, c, and d have degree 2). It is easy to see that R^G is Gorenstein in this example; more generally, by a result of Watanabe (Bruns and Herzog, 1993, Theorem 6.4.9), R^G is Gorenstein when $\rho(G) \subset SL_n$.

Remark 9. Since G is a finite group, the image of $\Psi : G \times V \to V \times V$ is a union of linear spaces (a subspace arrangement), one for each group element. Each of these linear spaces is cut out by an ideal $(y_1 - g \cdot x_1, \ldots, y_n - g \cdot x_n)$. With this explicit description it is possible to use syzygies to intersect these ideals to obtain the ideal for $\mathbb{I}(Y)$ rather than performing the elimination computation (see Vasconcelos (1998, page 29)). Unfortunately, the intersection computation requires the syzygies on $n|G|+1$ vectors, each of length n. Unless $|G|$ is very small, this is likely to be just as difficult as eliminating a single parameter, as required by Derksen's algorithm.

An algorithm to describe $D(R^G)$: We have treated all the tools that go into an algorithm to compute *generators* $a_1, \ldots, a_s$ for $D(R^G)$. In one sense, this would be sufficient to describe $D(R^G)$ since the relations among the generators are all induced from the relations $[x_i, \partial_j] = \delta_{ij}$ on the Weyl algebra. However, we will go further and describe the *relations* on $D(R^G)$ explicitly.

To this end, we may suppose that G contains no pseudoreflections (else replace R by R^W and G by G/W; however, note that when the resulting action is not linear we may need to use more sophisticated versions of Derksen's algorithm to compute generators for $D(R^W)^{G/W} = D(R)^G$) and so $D(R^G) = D(R)^G$. Suppose that we have found generators $a_1, \ldots, a_s$ for $D(R^G)$. We aim to compute the two-sided ideal of relations among the a_i. Of course, there are the commutator relations expressing $[a_i, a_j]$ in terms of the generators a_i. Introduce noncommuting variables $y_1, \ldots, y_s$, one for each generator a_i. Rewriting the commutators in terms of the generators $[a_i, a_j] = Q_{ij}(\mathbf{a})$ yields the commutator relations $y_i y_j = y_j y_i + Q_{ij}(\mathbf{y})$. Using commutators allows us to write all the remaining relations among the a_i as sums of scalar multiples of the ordered monomials $\mathbf{a}^{\mathbf{k}} = a_1^{k_1} a_2^{k_2} \cdots a_s^{k_s}$. Call an expression

$$T(\mathbf{y}) = T(y_1, \ldots, y_s) = \sum m_{\mathbf{k}} y_1^{k_1} y_2^{k_2} \cdots y_s^{k_s} = \sum m_{\mathbf{k}} \mathbf{y}^{\mathbf{k}} \tag{1}$$

an *ordered relation* if $T(\mathbf{a}) = 0$ in $D(R^G)$. Applying the symbol map gives $\sigma(T(\mathbf{a})) = \sum m_k \bar{\mathbf{a}}^{\mathbf{k}} = 0$ in $(GrD(R))^G \subset GrD(R)$, where the sum is now over those exponents $\mathbf{k}$ such that order$(\mathbf{a}^{\mathbf{k}})$ is maximal among terms in the original relation. Replacing $\bar{a}_i$ in this sum by $\bar{y}_i \in \mathbb{C}[\bar{y}_1, \ldots, \bar{y}_s]$ we get a polynomial $S(\bar{\mathbf{y}})$ with $S(\bar{\mathbf{a}}) = 0$. That is, $S(\bar{\mathbf{y}})$ is an element of the ideal

$$(\bar{y}_1 - \bar{a}_1, \ldots, \bar{y}_s - \bar{a}_s)\mathbb{C}[x_1, \ldots, x_n, \bar{\partial}_1, \ldots, \bar{\partial}_n, \bar{y}_1, \ldots, \bar{y}_s] \cap \mathbb{C}[\bar{y}_1, \ldots, \bar{y}_s].$$

Now let $\bar{p}_1(\bar{\mathbf{y}}), \ldots, \bar{p}_r(\bar{\mathbf{y}})$ be generators for this ideal. Replacing each $\bar{y}_i$ in $\bar{p}_j$ with a_i gives rise to an ordered polynomial in the a_i's that have smaller than expected order. This can be expressed as an ordered polynomial in the a_i's of lower degree than the degree of $\bar{p}_j$. This produces ordered relations $p_1(\mathbf{a}) = 0, \ldots, p_r(\mathbf{a}) = 0$ on the generators of $D(R^G)$ that extend the $\bar{p}_j$. Now if $S(\bar{\mathbf{y}}) := \sigma(T(\bar{\mathbf{y}})) = q_1(\bar{\mathbf{y}})\bar{p}_1(\bar{\mathbf{y}}) + \cdots + q_r(\bar{\mathbf{y}})\bar{p}_r(\bar{\mathbf{y}})$ then

$$T_{\text{new}}(\mathbf{a}) := T(\mathbf{a}) - (q_1(\mathbf{a})p_1(\mathbf{a}) + \cdots + q_r(\mathbf{a})p_r(\mathbf{a}))$$

is a relation in which the term of largest order is strictly smaller than the term of largest order in $T(\mathbf{a})$. Using the commutator relations, we can rewrite $T_{\text{new}}(\mathbf{a})$ as a sum of terms of smaller order than the largest term in T. Iteratively applying this procedure to $T_{\text{new}}(\mathbf{y})$, we see that the original relation $T(\mathbf{y})$ was in the two-sided ideal of the free algebra $\mathbb{C}\langle y_1, \ldots, y_s\rangle$ generated by the commutator relations and the ordered relations $p_1(\mathbf{y}), \ldots, p_r(\mathbf{y})$.

Algorithm 10. Given: A finite group G encoded as the common zero set of $f_1, \ldots, f_t \in \mathbb{C}[t_1, \ldots, t_d]$; a representation ρ of G on an n-dimensional complex vector space V (or the

induced action of G on $R = \mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_n]$).
Output: Explicit generators for $D(R^G)$ as elements of $\mathrm{End}_{\mathbb{C}}(R^G)$. A presentation of the ring $D(R^G)$ in terms of generators and relations.
Steps: (1) If G contains pseudoreflections, then let W be the subgroup they generate and replace R by R^W and G by G/W.
(2) The representation ρ induces a representation $\tilde{\rho} = \rho \oplus (\rho^{\mathrm{T}})^{-1}$ of G on $GrD(R)$. Obtain generators of $Gr(D(R)^G) = (GrD(R))^G$ using any algorithm for computing invariants of finite groups (for example, Derksen's algorithm).
(3) Lift these generators to $D(R)^G$ and apply π_* to obtain generators $a_1, \ldots, a_s$ of $D(R^G)$.
(4) Compute the commutator relations $[a_i, a_j]$ in $D(R^G) = D(R)^G$, expressing the commutators explicitly in terms of the a_i's. This gives rise to the commutator relations $y_i y_j = y_j y_i + Q_{ij}(\mathbf{y})$.
(5) Obtain ordered relations among the a_i's. Do this by first computing the relations among the $\bar{a}_i$'s by finding generators for the ideal

$$(\bar{y}_1 - \bar{a}_1, \ldots, \bar{y}_s - \bar{a}_s)\mathbb{C}[x_1, \ldots, x_n, \bar{\partial}_1, \ldots, \bar{\partial}_n, \bar{y}_1, \ldots, \bar{y}_s] \cap \mathbb{C}[\bar{y}_1, \ldots, \bar{y}_s]$$

using an elimination order. Each of these induces an ordered relation $p_i(\mathbf{y})$ on $D(R^G)$.
(6) The ideal of all relations among the generators of $D(R^G)$ is a two-sided ideal in the free $\mathbb{C}$-algebra $\mathbb{C}\langle y_1, \ldots, y_s\rangle$; this ideal is generated by the commutator relations from step (4) and the ordered relations from step (5).

Remark 11. (a) In step (1), there is a simple description of the subgroup W. If $\rho(\mathbf{t})$ represents the action of G on V, then the subgroup W consists of elements in G that correspond to $\mathbf{t}$ with $\mathrm{rank}(1 - \rho(\mathbf{t})) \leq 1$. Thus W corresponds to the subvariety cut out by the 2×2 minors of the matrix $1 - \rho(\mathbf{t})$.
(b) In step (1) we can compute R^W using Derksen's algorithm, or using the approach in Section 3.

2. Two examples

Example 12. We describe the ring of differential operators $D(R^G)$ on the ring R^G from Example 8. We first compute $(GrD(R))^G$. As before, represent G as the zero set of $f_1(t) = t^2 - 1$. Choose an ordered basis $\{x_1, x_2, x_3, \bar{\partial}_1, \bar{\partial}_2, \bar{\partial}_3\}$ of the vector space $GrD(R)_1$ and define the representation $\tilde{\rho} : G \to \mathrm{End}_{\mathbb{C}}(GrD(R)_1)$ via $\tilde{\rho}(t) = \rho(t) \oplus (\rho(t)^{\mathrm{T}})^{-1} = \rho(t) \oplus \rho(t)$. We compute a Gröbner basis $\mathcal{B}$ of the ideal

$$\begin{pmatrix} t^2 - 1, & y_1 - tx_1, & y_2 - \frac{t+1}{2}x_2 - \frac{1-t}{2}x_3, & y_3 - \frac{1-t}{2}x_2 - \frac{t+1}{2}x_3, \\ y_4 - t\bar{\partial}_1, & y_5 - \frac{t+1}{2}\bar{\partial}_2 - \frac{1-t}{2}\bar{\partial}_3, & y_6 - \frac{1-t}{2}\bar{\partial}_2 - \frac{t+1}{2}\bar{\partial}_3 & \end{pmatrix}$$

in the ring $\mathbb{C}[t, x_1, x_2, x_3, \bar{\partial}_1, \bar{\partial}_2, \bar{\partial}_3, y_1, y_2, y_3, y_4, y_5, y_6]$ with a term order that eliminates the first variable t. Considering only those elements of $\mathcal{B}$ that do not involve t and setting the y_i's to zero, we see

$$I = (\bar{\partial}_2 + \bar{\partial}_3, x_2 + x_3, \bar{\partial}_3^{\,2}, \bar{\partial}_1\bar{\partial}_3, x_3\bar{\partial}_3, x_1\bar{\partial}_3, \bar{\partial}_1^{\,2}, x_3\bar{\partial}_1, x_1\bar{\partial}_1, x_3^2, x_1x_3, x_1^2).$$

Applying the Reynolds operator,

$$\mathcal{R}(f(x_1, x_2, x_3, \bar{\partial}_1, \bar{\partial}_2, \bar{\partial}_3)) = [f(x_1, x_2, x_3, \bar{\partial}_1, \bar{\partial}_2, \bar{\partial}_3) + f(-x_1, x_3, x_2, -\bar{\partial}_1, \bar{\partial}_3, \bar{\partial}_2)]/2,$$

to these generators gives algebra generators for $(GrD(R))^G = Gr(D(R)^G)$ and lifting gives algebra generators for $D(R)^G$: $D(R)^G$ is generated by the twelve operators

$$\left\{\begin{matrix} \partial_2+\partial_3, & x_2+x_3, & \partial_2^2+\partial_3^2, & \partial_1\partial_3-\partial_1\partial_2, & x_2\partial_2+x_3\partial_3, & x_1\partial_3-x_1\partial_2, \\ \partial_1^2, & x_3\partial_1-x_2\partial_1, & x_1\partial_1, & x_2^2+x_3^2, & x_1x_3-x_1x_2, & x_1^2 \end{matrix}\right\}.$$

Now since G contains no pseudoreflections, the map $\pi_* : D(R)^G \to D(R^G)$ is surjective and so these operators, restricted to R^G, also generate $D(R^G)$.

There are many relations among the generators of $D(R^G)$. Label the 12 generators listed above $a_1, \ldots, a_{12}$. Table 1 lists the commutator relations among these operators. The commutator $[a_i, a_j]$ appears as the jth entry in the ith row of the table. A few entries have been labeled for compactness: $\alpha_{3,10} = 4a_5 + 4$, $\alpha_{4,6} = 2a_3 - a_1^2$, $\alpha_{4,11} = 2a_5 + 2a_9 - a_2a_1 + 2$, $\alpha_{6,8} = a_2a_1 + 2a_9 - 2a_5$, $\alpha_{7,12} = 4a_9 + 2$, $\alpha_{8,11} = 2a_{10} - a_2^2$.

Table 1
The commutators $[a_i, a_j]$ in $D(R^G)$

0	2	0	0	a_1	0	0	0	0	$2a_2$	0	0
-2	0	$-2a_1$	0	$-a_2$	0	0	0	0	0	0	0
0	$2a_1$	0	0	$2a_3$	0	0	$-2a_4$	0	$\alpha_{3,10}$	$2a_6$	0
0	0	0	0	a_4	$\alpha_{4,6}$	0	a_7	a_4	$2a_8$	$\alpha_{4,11}$	$2a_6$
$-a_1$	a_2	$-2a_3$	$-a_4$	0	$-a_6$	0	a_8	0	$2a_{10}$	a_{11}	0
0	0	0	$-\alpha_{6,4}$	a_6	0	$-2a_4$	$\alpha_{6,8}$	$-a_6$	$2a_{11}$	$2a_{12}$	0
0	0	0	0	0	$2a_4$	0	0	$2a_7$	0	$2a_8$	$\alpha_{7,12}$
0	0	$-2a_4$	$-a_7$	$-a_8$	$-\alpha_{6,8}$	0	0	a_8	0	$\alpha_{8,11}$	$2a_{11}$
0	0	0	$-a_4$	0	a_6	$-2a_7$	$-a_8$	0	0	a_{11}	$2a_{12}$
$-2a_2$	0	$-\alpha_{3,10}$	$-2a_8$	$-2a_{10}$	$-2a_{11}$	0	0	0	0	0	0
0	0	$-2a_6$	$-\alpha_{4,11}$	$-a_{11}$	$-2a_{11}$	$-2a_8$	$-\alpha_{8,11}$	$-a_{11}$	0	0	0
0	0	0	$-2a_6$	0	0	$-\alpha_{7,12}$	$-2a_{11}$	$-2a_{12}$	0	0	0

The elimination computation

$$(\bar{y}_1 - \bar{a}_1, \ldots, \bar{y}_s - \bar{a}_s)\mathbb{C}[x_1, \ldots, x_n, \bar{\partial}_1, \ldots, \bar{\partial}_n, \bar{y}_1, \ldots, \bar{y}_s] \cap \mathbb{C}[\bar{y}_1, \ldots, \bar{y}_s]$$

produces 40 generators $\bar{p}_1(\bar{\mathbf{y}}), \ldots, \bar{p}_{40}(\bar{\mathbf{y}})$ for the resulting ideal. Each of these relations $\bar{p}_i(\bar{\mathbf{y}})$ extends to a relation $p_i(\mathbf{y})$ on $D(R^G)$ given in Table 2.

The ideal of all relations in $D(R^G)$ is a two-sided ideal of $\mathbb{C}\langle y_1, \ldots, y_{12}\rangle$ generated by the commutator relations in Table 1 and the ordered relations in Table 2.

It is interesting to note that the Fourier transform (the map $F : D(R) \to D(R)$ that sends ∂_i to x_i and x_i to ∂_i but transposing all products) preserves the set of generators and the ideal of relations. Indeed, $F(a_1) = a_2$, $F(a_3) = a_{10}$, $F(a_4) = a_{11}$, $F(a_5) = a_5$, $F(a_6) = a_8$, $F(a_7) = a_{12}$ and $F(a_9) = a_9$. The fact that the Fourier transform acts on $D(R^G)$ in this example is due to the fact that G acts on both the x's and the ∂'s in the same way. In general, the Fourier transform will continue to act on R^G whenever the representation of G satisfies $\rho(G) \subset O(n)$, since then $(\rho(g)^{\mathrm{T}})^{-1} = \rho(g)$ and the action is the same on both x_i and ∂_i.

In this example we found a generating set of operators that were bi-homogeneous in the sense that when they are written in the form $\sum x^{\mathbf{d}}\partial^{\mathbf{e}}$, then all the $|d| = d_1 + \cdots + d_n$ have the same value and all the exponents $|e|$ are equal. This is a result that holds generally since the G-action preserves the bi-grading (total degree in the x's and total degree in the $\bar{\partial}$'s) on $GrD(R)$. The map π_* is just a restriction, so it does not affect the bi-homogeneity.

This example satisfies a conjecture of Singh (1986) regarding the differential operators on hypersurfaces. The ring R^G is a hypersurface (its defining equation is given in Example 8) and Singh conjectured that all singular hypersurfaces have order 2 differential operators

Table 2
Table of ordered relations

$a_9a_{11} - a_8a_{12} + a_{11}$	$a_6a_7 - a_4a_9 + a_4$
$a_8a_9 - a_7a_{11} + a_8$	$a_6a_9 - a_4a_{12} + 2a_6$
$a_9^2 - a_7a_{12} + 3a_9 + 2$	$-a_1a_2 + a_6a_8 - a_4a_{11} + 2a_5 + 4$
$a_2^2a_{12} + a_{11}^2 - 2a_{10}a_{12}$	$a_1a_2a_{12} + a_6a_{11} - 2a_5a_{12} - 4a_{12}$
$a_1^2a_{12} + a_6^2 - 2a_3a_{12}$	$a_7a_{11}^2 - a_8^2a_{12}$
$a_2a_6a_{11} - a_1a_{11}^2 - 2a_2a_5a_{12} + 2a_1a_{10}a_{12} - 6a_2a_{12}$	$a_6^2a_{10} - 2a_5a_6a_{11} + a_3a_{11}^2 + 6a_1a_2a_{12} + 2a_5^2a_{12} - 2a_3a_{10}a_{12} - 2a_5a_{12} - 12a_{12}$
$2a_2a_5a_9 - 2a_1a_9a_{10} - a_2a_4a_{11} + a_1a_8a_{11} + 2a_2a_5 + 6a_2a_9 - 2a_1a_{10} + 6a_2$	$a_1a_2a_9 + a_1a_2 - 2a_5a_9 + a_4a_{11} - 2a_5 - 4a_9 - 4$
$a_2^2a_9 + a_2^2 - 2a_9a_{10} + a_8a_{11} - 2a_{10}$	$a_1^2a_9 + a_1^2 + a_4a_6 - 2a_3a_9 - 2a_3$
$a_2a_4a_5 - a_2a_3a_8 + a_1a_5a_8 - a_1a_4a_{10} + 3a_2a_4$	$a_2a_5a_7 - a_2a_4a_8 + a_1a_8^2 - 2a_1a_7a_{10} + 6a_2a_7$
$a_2^2a_7 + a_8^2 - 2a_7a_{10}$	$a_1a_2a_7 - 2a_5a_7 + a_4a_8 - 4a_7$
$a_1^2a_7 + a_4^2 - 2a_3a_7$	$a_2a_6^2 - a_1a_6a_{11} - 2a_2a_3a_{12} + 2a_1a_5a_{12}$
$a_2a_5a_6 - a_1a_6a_{10} - a_2a_3a_{11} + a_1a_5a_{11} + 4a_2a_6$	$a_2a_4a_6 - 2a_2a_3a_9 + 2a_1a_5a_9 - a_1a_4a_{11} - 2a_2a_3 + 2a_1a_5$
$a_2^2a_6 - a_1a_2a_{11} - 2a_6a_{10} + 2a_5a_{11} + 4a_{11}$	$a_1a_2a_6 - a_1^2a_{11} - 2a_5a_6 + 2a_3a_{11} - 6a_6$
$2a_5^2a_7 - 2a_4a_5a_8 + a_3a_8^2 + a_4^2a_{10} - 2a_3a_7a_{10} + 10a_5a_7 - 4a_4a_8 + 12a_7$	$a_2a_4a_6 - 2a_2a_3a_9 + 2a_1a_5a_9 - a_1a_4a_{11} - 2a_2a_3 + 2a_1a_5$
$a_2^2a_4 - a_1a_2a_8 + 2a_5a_8 - 2a_4a_{10} + 4a_8$	$a_1a_2a_4 - a_1^2a_8 - 2a_4a_5 + 2a_3a_8 - 4a_4$
$a_2^2a_3 - 2a_1a_2a_5 + a_1^2a_{10} - 2a_1a_2 + 2a_5^2 - 2a_3a_{10} + 10a_5 + 8$	$a_2a_4a_{11}^2 - a_1a_8a_{11}^2 - 2a_2a_5a_8a_{12} + 2a_1a_8a_{10}a_{12} - 6a_2a_8a_{12}$
$-2a_2^2a_3 + 4a_1a_2a_5 + 4a_1a_2a_9 + 2a_5^2a_9 - 2a_1^2a_{10} + a_4a_6a_{10} - 2a_3a_9a_{10} - 2a_4a_5a_{11} + a_3a_8a_{11} + 8a_1a_2 - 2a_5^2 + 2a_5a_9 + 2a_3a_{10} - 18a_5 - 4a_9 - 20$	$a_4a_6a_{10}a_{11} - 2a_4a_5a_{11}^2 + a_3a_8a_{11}^2 + 2a_5^2a_8a_{12} - 2a_3a_8a_{10}a_{12} - 4a_4a_{11}^2 + 10a_5a_8a_{12} + 12a_8a_{12}$
$a_1a_2a_3a_8 - 2a_1^2a_5a_8 + a_1^2a_4a_{10} - 2a_4a_5^2 - 2a_1^2a_8 + 2a_3a_5a_8 - 10a_4a_5 + 4a_3a_8 - 12a_4$	$a_1^2a_6a_{10} + a_1a_2a_3a_{11} - 2a_1^2a_5a_{11} - 2a_5^2a_6 - 2a_1^2a_{11} + 2a_3a_5a_{11} - 14a_5a_6 + 4a_3a_{11} - 24a_6$
$a_1a_6a_{10}a_{11} + a_2a_3a_{11}^2 - 2a_1a_5a_{11}^2 - 2a_2a_5^2a_{12} + 2a_1a_5a_{10}a_{12} - 2a_1a_{11}^2 - 14a_2a_5a_{12} + 4a_1a_{10}a_{12} - 24a_2a_{12}$	$2a_4a_5a_8a_{11}^2 - a_3a_8^2a_{11}^2 - a_4^2a_{10}a_{11}^2 - 2a_5^2a_8^2a_{12} + 2a_3a_8^2a_{10}a_{12} + 4a_4a_8a_{11}^2 - 10a_5a_8^2a_{12} - 12a_8^2a_{12}$
$a_2a_3a_8a_{11}^2 - 2a_1a_5a_8a_{11}^2 + a_1a_4a_{10}a_{11}^2 - 2a_2a_5^2a_8a_{12} + 2a_1a_5a_8a_{10}a_{12} - 2a_1a_8a_{11}^2 - 14a_2a_5a_8a_{12} + 4a_1a_8a_{10}a_{12} - 24a_2a_8a_{12}$	$2a_1^2a_4a_5a_8 - a_1^2a_3a_8^2 - a_1^2a_4^2a_{10} + 2a_4^2a_5^2 + 4a_1^2a_4a_8 - 4a_3a_4a_5a_8 + 2a_3^2a_8^2 + 10a_4^2a_5 - 8a_3a_4a_8 + 12a_4^2$

that are not generated by derivations. Singh's conjecture is a strengthened form of Nakai's conjecture: all singular affine varieties have a differential operator that is not generated by the derivations.

Theorem 13 (*Ishibashi, 1985; Schwarz, 1995*). *Nakai's conjecture holds for all orbifolds: If G is a finite group acting on a polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ then $D(R^G)$ is generated as an R^G-algebra by derivations if and only if R^G is a polynomial ring.*

Of course, R^G is a polynomial ring precisely when G is a reflection group, by the Sheppard–Todd–Chevalley theorem. Though Nakai's conjecture holds for orbifolds, Singh's conjecture remains open for R^G.

We now treat an example in which G contains pseudoreflections but is not itself a reflection group.

Example 14. Let $G = \langle \gamma, \tau, \delta : \gamma^2 = \tau^2 = \delta^2 = 1,\ \gamma\delta = \tau\gamma, \tau\delta = \delta\tau \rangle$. The group G has 8 elements and if $V = \mathbb{C}^3$, G can be represented by $\rho : G \to GL(V)$ satisfying

$$\rho(\gamma) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \rho(\tau) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \rho(\delta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Letting $R = \mathbb{C}[V]$, we will compute $D(R^G)$.

G contains a normal subgroup generated by pseudoreflections, $W = \langle \tau, \delta \rangle$. The ring of invariants $R^W = \mathbb{C}[x_1, x_2^2, x_3^2]$ can be computed by inspection. Set $z_1 = x_1$, $z_2 = x_2^2$ and $z_3 = x_3^2$ so that R^W is the polynomial ring $\mathbb{C}[z_1, z_2, z_3]$. The group $G/W = \{e + W, \gamma + W\}$ acts on R^W. Of course $e + W$ acts as the identity and $\gamma + W$ acts via the matrix $\rho(\gamma)$. This is precisely the situation examined in Examples 8 and 12. Using those computations gives

$$R^G = \mathbb{C}[z_2 + z_3, z_2^2 + z_3^2, z_1 z_3 - z_1 z_2, z_1^2] = \mathbb{C}[x_2^2 + x_3^2, x_2^4 + x_3^4, x_1 x_3^2 - x_1 x_2^2, x_1^2]$$

and $D(R^G)$ is generated by the 12 operators

$$\left\{ \begin{matrix} \partial_2 + \partial_3, & z_2 + z_3, & \partial_2^2 + \partial_3^2, & \partial_1\partial_3 - \partial_1\partial_2, & z_2\partial_2 + z_3\partial_3, & z_1\partial_3 - z_1\partial_2, \\ \partial_1^2, & z_3\partial_1 - z_2\partial_1, & z_1\partial_1, & z_2^2 + z_3^2, & z_1 z_3 - z_1 z_2, & z_1^2 \end{matrix} \right\},$$

where now ∂_i stands for $\partial/\partial z_i$. The ring $D(R^G)$ cannot be generated by operators expressed in terms of the $\partial/\partial x_i$.

3. An alternate method to compute $(\mathrm{GrD(R)})^{\mathrm{G}}$

It is also possible to use the Reynolds operator and the Molien series to describe rings of invariants like R^W and $(GrD(R))^G$. In general, if G is a group acting on $R = \mathbb{C}[x_1, \ldots, x_n]$, then the Molien series is the Hilbert series for the ring R^G, a series that encodes the dimensions of the graded pieces of R^G:

$$H(R^G, t) = \sum_{d=0}^{\infty} \left(\dim_{\mathbb{C}} R_d^G \right) t^d.$$

In 1897 Molien proved that it is possible to compute $H(R^G, t)$ without first computing R^G (Sturmfels, 1993, Theorem 2.2.1).

Theorem 15 (*Molien's Theorem*). *If G is a finite group of order $|G|$ acting on $R = \mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_n]$ via the representation $\rho : G \to GL(V)$ then the Molien series can be expressed as*

$$H(R^G, t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - t\rho(g))}.$$

Hochster and Roberts proved that if G is a linearly reductive group acting on $R = \mathbb{C}[V]$ then R^G is Cohen–Macaulay (Hochster and Roberts, 1974). In particular, when G is finite then $R^G = \mathbb{C}[V]^G$ is Cohen–Macaulay (this seems to have been first written down by Hochster and Eagon (1971, Proposition 13); for an elementary proof see Stanley (1979, Theorem 3.2)). In particular, there is a homogeneous system of parameters $h_1, \ldots, h_n$ in R^G and R^G is a free module over the polynomial ring $\mathbb{C}[h_1, \ldots, h_n]$. The invariants $h_1, \ldots, h_n$ are called primary invariants and secondary invariants are given by any $k[h_1, \ldots, h_n]$-module basis for R^G. The Molien series can then be written in the form

$$H(R^G, t) = \frac{p(t)}{\prod_{i=1}^{n}(1 - t^{\deg(h_i)})}.$$

The degrees of the primary invariants can be read off this expression, as can the degrees d_i and number in each degree m_i of the secondary invariants: these are encoded by the polynomial $p(t) = \sum m_i t^{d_i}$. There are good algorithms to compute the primary invariants (see Decker et al. (1998)). Once these are found, we can apply the Reynolds operator to a basis for R_d until the results (together with $R_d^G \cap \mathbb{C}[h_1, \ldots, h_n]$) span a vector space of dimension $\dim_{\mathbb{C}}(R_d^G)$, as predicted by the Molien series.

This method can be applied to the ring $(GrD(R))^G$, but we need to use the grading given by total degree so that the graded pieces are all finite dimensional.

Example 16. We apply the alternate method to Example 12. The Molien series for $(GrD(R))^G$ is

$$\frac{1}{|G|}\sum_{g\in\{-1,1\}}\frac{1}{\det(1 - t\tilde{\rho}(g))} = \frac{1}{2}\left[\frac{1}{(1-t)^6} + \frac{1}{(1+t)^4(1-t)^2}\right] = \frac{1 + 6t^2 + t^4}{(1-t)^2(1-t^2)^4}.$$

This suggests that there is a homogeneous system of parameters of $GrD(R)$ consisting of six generators, two with total degree 1 and four with total degree 2. One such system consists of the primary invariants $x_2 + x_3$, $\bar{\partial}_2 + \bar{\partial}_3$, x_1^2, $\bar{\partial}_1^2$, $x_2^2 + x_3^2$, and $\bar{\partial}_2^2 + \bar{\partial}_3^2$. The vector space $(GrD(R))_2^G$ of degree 2 elements is the image of the Reynolds operator applied to $(GrD(R))_2$. Modulo the subspace generated by all degree 2 polynomials in the primary invariants, $(GrD(R))_2^G$ has dimension 6. One basis for this space is $\{x_1x_3 - x_1x_2, x_1\bar{\partial}_1, x_1\bar{\partial}_2 - x_1\bar{\partial}_3, x_2\bar{\partial}_1 - x_3\bar{\partial}_1, x_2\bar{\partial}_2 + x_3\bar{\partial}_3, \bar{\partial}_1\bar{\partial}_3 - \bar{\partial}_1\bar{\partial}_2\}$. Together with the primary invariants, these secondary invariants generate $(GrD(R))^G$ *as a* $\mathbb{C}$*-algebra* (see Example 12). However, these six secondary invariants do not generate $(GrD(R))^G$ *as a module* over the polynomial ring generated by the primary invariants. Indeed, the Molien series indicates that a degree 4 secondary invariant is required to generate $(GrD(R))^G$ as a module over the ring generated by the primary invariants. For instance, $\mathcal{R}(x_1x_2\bar{\partial}_1\bar{\partial}_2) = \frac{1}{2}(x_1x_2\bar{\partial}_1\bar{\partial}_2 + x_1x_3\bar{\partial}_1\bar{\partial}_3)$ is one such secondary invariant. It turns out that if we are only interested in obtaining generators of $D(R)^G$, rather than generators whose symbols also generate $(GrD(R))^G$, then it always suffices to use the primary invariants generating $\mathbb{C}[V]^G$ and $\mathbb{C}[V^*]^G$ (see Levasseur and Stafford (1995, Theorem 5)).

It is possible to refine Molien's Theorem to the bi-graded case. This allows us to more easily locate the secondary invariants.

Theorem 17. *Let V and W be complex vector spaces of dimensions m and n. Also, let G be a finite group of order $|G|$ acting on $R = \mathbb{C}[V \oplus W] = \mathbb{C}[x_1, \ldots, x_m, y_1, \ldots, y_n]$ via the*

representation $\rho : G \to GL(V) \times GL(W)$. Then R^G is a bi-graded ring with the bi-grading given by total degree in the x's and total degree in the y's. The bi-graded Molien series can be expressed as

$$H(R^G, s, t) := \sum_{d,e} \left(\dim_{\mathbb{C}} R^G_{d,e}\right) s^d t^e = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - s\rho(g)|_V)\det(1 - t\rho(g)|_W)}.$$

We omit the proof; it suffices to generalize Sturmfels's proof of Molien's Theorem from (Sturmfels, 1993, Theorem 2.2.1) to the bi-graded case.

Example 18. The bi-graded Molien series for $(GrD(R))^G$ from Example 16 is

$$\frac{1}{2}\left[\frac{1}{(1-s)^3(1-t)^3} + \frac{1}{(1+s)^2(1-s)(1+t)^2(1-t)}\right]$$
$$= \frac{1 + s^2 + 4st + t^2 + s^2t^2}{(1-s)(1-t)(1-s^2)^2(1-t^2)^2}.$$

The homogeneous system of parameters was graded in bi-degrees (1, 0), (0, 1), (2, 0) and (0, 2), as this formula predicts. As well, the secondary invariants were located in bi-degrees (2, 0), (1, 1), (0, 2) and (2, 2). Using the bi-graded version of Molien's Theorem to search for these secondary invariants greatly reduces the size of the vector spaces to which we need to apply the Reynolds operator since we can work with a bi-graded piece $R_{d,e}$ rather than all of R_n.

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