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Differential Algebras on Semigroup Algebras

Mutsumi Saito and William N. Traves

Abstract. This paper studies algebras of operators associated to a semigroup algebra. The ring of differential operators is shown to be anti-isomorphic to the symmetry algebra and both are described explicitly in terms of the semigroup. As an application, we produce a criterion to determine the equivalence of A-hypergeometric systems. Conditions under which associated algebras are finitely generated are studied. These results are sufficient to establish Becker's conjecture in the semigroup case. As well, an algorithm is provided to compute the composition series of D-modules over semigroup algebras.

1. Introduction

This paper studies two algebras of operators associated to a semigroup algebra. The ring of differential operators was first introduced by Sweedler [**20**] and Grothendieck [**4**] and has been extensively studied in the context of normal toric varieties [**8, 12, 13**]. The symmetry algebra is a more recent arrival connected with the study of A-hypergeometric differential equations. These systems of partial differential equations are determined by two parameters, a semigroup and a vector. The symmetry algebra was introduced by Saito [**14**] to characterize parameters giving rise to equivalent systems.

We begin the paper by telling "a tale of two algebras." Both authors presented papers about differential algebras on toric varieties at the AMS-IMS-SIAM Summer research conference at Mount Holyoke in 2000. The operators appearing in Theorem 3.2.2 appeared in both talks, prompting our collaboration. Indeed, while at the conference we proved Theorem 2.3.3, showing that the symmetry algebra and the ring of differential operators are anti-isomorphic.

Both algebras of operators are introduced in Section 2 in great generality; however, we confine our attention to semigroup algebras in the remainder of the paper. We begin Section 3 with an explicit description of the ring of differential

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operators $D(R_A)$ on a normal toric variety due to Jones [**8**]. In this case, the graded ring $GrD(R_A)$ is finitely generated. In contrast, we give an example showing that $GrD(R_A)$ need not be finitely generated if R_A is a nonnormal semigroup algebra. This example reappears throughout the paper; we use it to illustrate many of our constructions.

We introduce the notion of a scored semigroup algebra in order to characterize those semigroups $\mathbb{N}A$ with $GrD(\mathbb{C}[\mathbb{N}A])$ finitely generated. Indeed, we conjecture that all rings of differential operators on affine semigroups are finitely generated, but only the scored semigroups admit finitely generated graded algebras $GrD(\mathbb{C}[\mathbb{N}A])$. Evidence is provided to support this conjecture.

We then turn our attention to the map $D(R) \to D(R, R/m)$ from the ring of differential operators to the module of constant coefficient differential operators. We show that Becker's conjecture holds for semigroup algebras: when $GrD(R)$ is finitely generated, the map $D(R) \to D(R, R/m)$ is surjective. One of the key ingredients in the proof is an explicit characterization of $D(R, R/m)$ in the case of semigroup algebras.

Section 3 closes with an algorithm to determine the graded pieces of the ring of differential operators on a semigroup algebra. This is heavily dependent on the computational insights developed in [**15**]. As an application of our anti-isomorphism, the theorem determining the equivalence of A-hypergeometric systems is extended to non-homogeneous systems.

In the final section we study the structure of modules over the ring of differential operators on a semigroup algebra. Using the algorithms developed in Section 3, we develop an algorithm to determine the composition series of such a module. In turn, this leads to a classification of those semigroup algebras R_A that are $D(R_A)$-simple. An example is given to illustrate the contrast with the situation for saturated semigroups: all normal toric varieties are D-simple.

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2. A Tale of Two Algebras (of Differential Operators)

2.1. Rings of Differential Operators. There are many equivalent definitions of the ring of differential operators on an algebraic variety. Here, we present the most elementary and best-motivated definition. To fix notation, let X be an affine algebraic subvariety of $\mathbb{C}^n$. Let R be the coordinate ring of X: $X = \mathrm{Spec}(R)$. We write $X = \mathbb{V}(I)$ and $R = \frac{\mathbb{C}[x_1,\ldots,x_n]}{I}$.

Just as we define the ring of functions on X by restricting functions on the ambient space to X, we would also like to realize differential operators on X by restricting operators in the Weyl algebra, $W = \mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$, to R (here ∂_i stands for $\frac{\partial}{\partial x_i}$). Of course, not every operator in the Weyl algebra acts on R in a well-defined way: only the operators in the idealizer of I, $\{\theta \in W : \theta(I) \subseteq I\}$, act on R. Further, we quotient the idealizer by the set of operators whose image lies entirely in I. It is easy to check that these are just the operators in IW. This motivates the following description of differential operators on subvarieties of affine space.

Definition 2.1.1. Let $R = \frac{\mathbb{C}[x_1,\dots,x_n]}{I}$. The ring of differential operators on R (or on $\mathrm{Spec}(R)$) is

$$D(R) = \frac{\{\theta \in W : \ \theta * I \subseteq I\}}{IW},$$

here, as in the rest of the paper, we write $\theta * \mathbf{b}$ to mean the image of $\mathbf{b} \in R$ under the map θ.

This definition is equivalent to more abstract definitions such as: Grothendieck's definition in terms of commutators ([**4**]), the definition in terms of the action of the enveloping algebra $R \otimes_{\mathbb{C}} R$ on $\mathrm{End}_{\mathbb{C}}(R)$ ([**18**]) and the realization of $D(R)$ as the endomorphisms of R that are continuous in every I-adic topology on R (for all ideals I of R; see [**2**]).

2.2. Symmetry Algebra. The symmetry algebra of A-hypergeometric systems was introduced in [**14**] to study morphisms among A-hypergeometric systems with different parameters. Here we define the symmetry algebra in a wider context.

Let $\rho : G \to GL(V)$ be a representation of a complex linear algebraic group G on $V = \mathbb{C}^n$. Let I be a G-stable left ideal of W. Given a character λ of the Lie algebra $\mathrm{Lie}(G)$, we define a W-module

$$M_I(\lambda) = W/(I + \sum_{\xi \in \mathrm{Lie}(G)} W(L_\xi - \lambda(\xi))),$$

where L_ξ is the vector field on $V = \mathbb{C}^n$ induced from the G-action: $(L_\xi f)(v) = \frac{d}{dt} f(e^{-t\xi} v)_{|t=0}$. Let $d\rho^*$ denote the differential of the contragradient action on V^*, and also its representation matrix with respect to the basis $x_1, \dots, x_n$. Then

$$L_\xi = -\sum_{i,j} d\rho^*(\xi)_{ij} x_j \partial_i. \tag{1}$$

The module $M_I(\lambda)$ is trivial unless λ vanishes on $\mathrm{Ker}(d\rho^*)$. Hence it is natural to assume that the representation $d\rho^*$ is faithful, and we do so from now on.

Example 2.2.1. Let $\mathbf{a}_j = {}^t(a_{1j}, \dots, a_{dj}) \in \mathbb{Z}^d$ be the jth column of the matrix A ($j = 1, \dots, n$), and let $T = \{\, (t_1, \dots, t_d) \,|\, t_1, \dots, t_d \in \mathbb{C}^\times \,\}$ act on $V = \mathbb{C}^n$ via

$$\mathbf{t} \cdot \mathbf{v} = (t_1, \dots, t_d) \cdot (v_1, \dots, v_d) = (t^{a_1} v_1, \dots, t^{a_n} v_n),$$

where $t^{\mathbf{a}_j} = t_1^{a_{1j}} \cdots t_d^{a_{dj}}$. Let $\mathbf{e}_1, \dots, \mathbf{e}_d$ denote the standard basis of $\mathrm{Lie}(T) = \mathbb{C}^d$. Then $d\rho^*(\mathbf{e}_i) = \mathrm{diag}(-a_{i1}, \dots, -a_{in})$, and thus $L_{\mathbf{e}_i} = \sum_{j=1}^n a_{ij} x_j \partial_j$. In this case, $d\rho^*$ is faithful if and only if the matrix $A = (\mathbf{a}_1, \dots, \mathbf{a}_n)$ has rank d.

The most intriguing case occurs when $I = W\mathbb{I}(X)$ for a G-stable subvariety X of the dual space V^*, where $\mathbb{I}(X)$ is the defining ideal of X in $\mathbb{C}[V^*] = \mathbb{C}[\partial_1, \dots, \partial_n]$. This case includes many interesting examples, such as: the system for the relative invariants of regular prehomogeneous vector spaces; Harish-Chandra systems on Lie algebras (see [**7**]); A-hypergeometric systems (cf. Example 2.2.1) and their generalizations, Tanisaki's systems ([**21**]) and Kapranov's systems ([**9**]), which will be implicitly considered in the next section.

Definition 2.2.2. Let $\tilde{S}(W/I) := \{\, \theta \in W \,|\, I\theta \subset I \,\}$. Then the algebra $S(W/I) := \tilde{S}(W/I)/I = \mathrm{End}_W(W/I)$ is called the *symmetry algebra* of the W-modules $M_I(\lambda)$.

By differentiating the G-stability of I, we see $L_\xi \in \tilde{S}(W/I)$ for all $\xi \in \mathrm{Lie}(G)$. Given a character χ of $\mathrm{Lie}(G)$, put

$$S(W/I)_\chi := \{\, \theta \in S(W/I) \mid [L_\xi, \theta] = \chi(\xi)\theta \quad \text{for all } \xi \in \mathrm{Lie}(G) \,\}.$$

Remark 2.2.3. The operators in $S(W/I)_\chi$ are contiguity operators shifting 'parameters' by χ in the following sense. Let $\theta \in S(W/I)_\chi$, and ψ be a solution to $M_I(\lambda)$. Then $\theta(\psi)$ is a solution to $M_I(\lambda + \chi)$.

Since the action comes from the action of G, $S(W/I)_\chi = 0$ unless χ is the differential of a character of G. When G is an algebraic torus, we have

$$S(W/I) = \bigoplus_{\chi \in \mathrm{Hom}(G, \mathbb{C}^\times)} S(W/I)_{d\chi},$$

where $d\chi$ denotes the differential of a character χ of G.

2.3. Equivalence of the Two Algebras.

Lemma 2.3.1. *If I is an ideal of $R = \mathbb{C}[x_1, \ldots, x_n]$ and θ is an element of the Weyl algebra, $W = \mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$, then $\theta IW \subset IW$ if and only if $\theta * I \subset I$.*

Proof. Suppose that $\theta IW \subset IW$. Then $\theta \circ I \subset IW$ so

$$\theta * I = (\theta \circ I) * 1 \subset IW * 1 = I.$$

Conversely, suppose that $\theta * I \subset I$. Let η be an element of IW and write $\theta \circ \eta = \sum_{\mathbf{a}} P_{\mathbf{a}} \partial^{\mathbf{a}}$ where $\partial^{\mathbf{a}} = \frac{1}{a_1!}\partial_1^{a_1} \cdots \frac{1}{a_n!}\partial_n^{a_n}$. If each $P_{\mathbf{a}} \in I$ then $\theta \circ \eta \in IW$ and we are done. Aiming for a contradiction, assume that some $P_{\mathbf{a}} \notin I$. Let $\mathbf{b}$ be an n-tuple of minimal total degree such that $P_{\mathbf{b}} \notin I$. Then

$$(\theta \circ \eta) * x^{\mathbf{b}} \equiv P_{\mathbf{b}} \mod I.$$

Since $\theta * I \subset I$ and the image of the map $\eta : R \to R$ is in I, it follows that the image of $\theta \circ \eta$ is contained in I. So $P_{\mathbf{b}} \in I$, producing a contradiction. □

Let $\phi : W \to W$ denote the involutive anti-automorphism interchanging x_i and ∂_i for all i. Let $\rho : G \to GL(V)$ be a representation of a complex linear algebraic group G on $V = \mathbb{C}^n$.

Lemma 2.3.2. *If $\xi, \xi' \in \mathrm{Lie}(G)$ satisfy ${}^t d\rho^*(\xi) = d\rho^*(\xi')$, then $\phi(L_\xi) = L_{\xi'} + C$, where $C \in \mathbb{C}$ and ^{t} stands for the operation of taking the transposed matrix.*

Proof. This is immediate from the equation (1). □

If ${}^t d\rho^*(\mathrm{Lie}(G)) = d\rho^*(\mathrm{Lie}(G))$, then for each ξ there exists unique ξ' satisfying ${}^t d\rho^*(\xi) = d\rho^*(\xi')$, since the representation $d\rho^*$ is faithful. In this case we define $\phi(\xi)$ to be the element satisfying ${}^t d\rho^*(\xi) = d\rho^*(\phi(\xi))$. Clearly ϕ defines an involutive anti-automorphism of $\mathrm{Lie}(G)$.

Theorem 2.3.3. *Let $I(x)$ be a G-stable ideal of $\mathbb{C}[x_1, \ldots, x_n]$. Assume that ${}^t\rho(G) = \rho(G)$, and that the representation $d\rho^*$ is faithful. Then $I(\partial) := \phi(I(x))$ is G-stable, and ϕ induces an anti-isomorphism between the symmetry algebra $S(W/WI(\partial))$ and the ring of differential operators $D(\mathbb{C}[x]/I(x))$. The map ϕ respects but does not preserve the grading: the χ-graded piece of one algebra is sent to the $(-\chi \circ \phi)$-graded piece of the other algebra.*

Proof. Let $g, h \in G$ satisfy $\rho(g^{-1}) = {}^t\rho(h)$. Then we have $g.\phi(f) = \phi(h.f)$ for $f \in \mathbb{C}[x_1, \ldots, x_n]$. Hence $I(\partial)$ is G-stable. The next statement follows from Lemma 2.3.1 as

$$D(\mathbb{C}[x]/I) = \frac{\{\theta \in W : \theta I(x)W \subset I(x)W\}}{I(x)W}$$

and

$$S(W/WI(\partial)) = \frac{\{\theta \in W : I(\partial)\theta \subset WI(\partial)\}}{WI(\partial)}.$$

Note that if χ is a character of $\mathrm{Lie}(G)$, then so is $\chi \circ \phi$. The final result is established by a simple computation:

$$\theta \in S(W/WI(\partial))_\chi \iff [L_\xi, \theta] = L_\xi\theta - \theta L_\xi = \chi(\xi)\theta \ (\xi \in \mathrm{Lie}(G)).$$

Applying ϕ gives

$$\begin{array}{rrcll} & \phi(\theta)\phi(L_\xi) - \phi(L_\xi)\phi(\theta) & = & \chi(\xi)\phi(\theta) & (\xi \in \mathrm{Lie}(G)) \\ \iff & \phi(\theta)L_{\phi(\xi)} - L_{\phi(\xi)}\phi(\theta) & = & \chi(\xi)\phi(\theta) & (\xi \in \mathrm{Lie}(G)) \\ \iff & \phi(\theta)L_\xi - L_\xi\phi(\theta) & = & \chi(\phi(\xi))\phi(\theta) & (\xi \in \mathrm{Lie}(G)) \\ \iff & [L_\xi, \phi(\theta)] & = & -\chi \circ \phi(\xi)\phi(\theta) & (\xi \in \mathrm{Lie}(G)) \\ \iff & \phi(\theta) & \in & D(R)_{-\chi\circ\phi}. & \end{array}$$

□

3. Differential Algebras on Toric Varieties

3.1. Toric Varieties. Our description of toric varieties follows Sturmfels [**19**]. Let $A = (a_{ij})$ be a $d \times n$ matrix of full rank whose entries lie in $\mathbb{Z}$. The columns $\mathbf{a}_i$ of A generate a semigroup and we consider the semigroup algebra $R_A = \mathbb{C}[t^{\mathbf{a_1}}, \ldots, t^{\mathbf{a_n}}]$, where we have used multi-index notation: $t^{\mathbf{a_i}} = t_1^{a_{i1}} \cdots t_d^{a_{id}}$. The algebra R_A has a presentation given by the short-exact sequence,

$$0 \to I_A \to \mathbb{C}[x_1, \ldots, x_n] \to R_A \to 0,$$

and the ideal $I_A(x)$ is generated by binomials in the variables $x_1, \ldots, x_n$. Indeed, I_A is generated by terms $x^{\mathbf{p}} - x^{\mathbf{n}}$ where $\mathbf{p}$ and $\mathbf{n}$ are vectors of non-negative integers such that $A(\mathbf{p} - \mathbf{n}) = \mathbf{0}$. For later use, let $I_A(\partial)$ denote the similar ideal in the polynomial ring generated by the ∂_i's:

$$I_A(\partial) = (\{\partial^{\mathbf{p}} - \partial^{\mathbf{n}} : \ A(\mathbf{p} - \mathbf{n}) = \mathbf{0}\}) \subset \mathbb{C}[\partial_1, \ldots, \partial_n].$$

The saturation of the semigroup generated by the columns of A consists of a cone σ in a lattice $\mathbb{Z}A$ bounded by hyperplanes $h_i(t_1, \ldots, t_d) = 0$ (oriented so that the functionals h_i are positive on the cone σ). We determine each h_i uniquely by requiring $h_i(\mathbb{Z}A) = \mathbb{Z}$.

3.2. Ring of Differential Operators for Affine Toric Varieties. We begin with an explicit description of the ring of differential operators on an affine toric variety. Our description follows Jones [**8**] and Musson [**12**]. To start, we note that differential operators behave well under localization.

Lemma 3.2.1. *If S is a multiplicatively closed set in R, then*

$$S^{-1}R \otimes_R D(R) = D(S^{-1}R).$$

Proof. This follows from the observation that differential operators of bounded order are determined by maps to R from a universal object (the jet module; see Grothendieck [**4**]). □

This allows us to reduce to the case where the cone σ generated by the columns of A is strongly convex (it does not contain any lines through the origin). If the cone σ is not strongly convex, then σ contains a strongly convex cone $\mathbb{N}B \subset \sigma$ so that $R_A = \mathbb{C}[\mathbb{N}A]$ is a localization of $\mathbb{C}[\mathbb{N}B]$.

At this stage, $\mathbb{Z}A = n_1\mathbb{Z} \times \cdots \times n_d\mathbb{Z}$ for nonzero integers $n_1, \ldots, n_d$. After multiplying row i by $1/n_i$ (this does not change the isomorphism type of R_A), we may assume that the lattice $\mathbb{Z}A$ equals $\mathbb{Z}^d$ and A is a $d \times n$ integer matrix.

Now let S be the multiplicatively closed set $\mathbb{C}[\mathbb{N}A \setminus \mathbf{0}]$ in R_A. Then

$$S^{-1}R_A \cong \mathbb{C}[\mathbb{Z}A] = \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}].$$

Lemma 3.2.1 now implies that every differential operator on R_A can be realized as a differential operator on $\mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$:

$$D(R_A) \subset D(\mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]) = \mathbb{C}\langle t_1^{\pm 1}, \ldots, t_d^{\pm 1}, \partial_1, \ldots, \partial_d\rangle = \mathbb{C}\langle t_1^{\pm 1}, \ldots, t_d^{\pm 1}, \theta_1, \ldots, \theta_d\rangle,$$

where $\theta_i = t_i\partial_i$. The multigrading on $\mathbb{C}[\mathbb{Z}^d]$ induces a multigrading on $D(\mathbb{C}[\mathbb{Z}^d])$: $t^{\mathbf{a}}$ is assigned degree $\mathbf{a}$ and the θ_i are assigned degree $\mathbf{0}$. In turn, this induces a multidegree on $D(R_A)$. Consider the graded piece of $D(R_A)$ of degree $\mathbf{a}$. Each element of this module is a sum of elements of the form $t^{\mathbf{a}}P(\theta)$ where $P(\theta) = P(\theta_1, \ldots, \theta_d)$ is an operator in $\mathbb{C}[\mathbb{Z}^d]$ of multidegree $\mathbf{0}$. In order for the operator $t^{\mathbf{a}}P(\theta)$ to induce an action on R_A it must stabilize $\mathbb{C}[\mathbb{N}A]$. Furthermore, every operator in $D(R_A)_{\mathbf{a}}$ is obtained in this way. The operators $t^{\mathbf{a}}P(\theta)$ that stabilize $\mathbb{C}[\mathbb{N}A]$ are those operators in which $P(\theta)$ vanishes on the set

$$\Omega_{\mathbb{N}A}(\mathbf{a}) = \Omega(\mathbf{a}) = ZC(\{\mathbf{b} \in \mathbb{N}A : \ \mathbf{a} + \mathbf{b} \notin \mathbb{N}A\}),$$

where $ZC(S)$ indicates the closure of S in the Zariski topology on $\mathbb{Z}^d$; that is, we require that $P(\theta)$ lie in the ideal $\mathbb{I}(\Omega(\mathbf{a}))$ of $\mathbb{C}[\theta_1, \ldots, \theta_d]$. When the semigroup $\mathbb{N}A$ is saturated (that is, the algebra R_A is normal), this ideal is principal and the generator has a particularly nice description in terms of the boundary hyperplanes $h_i(\theta_1, \ldots, \theta_d) = 0$.

Theorem 3.2.2. *Let R_A be the coordinate ring of a normal toric variety whose associated semigroup $\mathbb{N}A$ is the saturated cone bounded by the hyperplanes $h_i = 0$ $(i = 1, \ldots, k)$. Then:*

(1) We have the following description of the ring of differential operators as a graded object:

$$D(R_A) = \bigoplus_{\mathbf{a} \in \mathbb{Z}A} D(R_A)_{\mathbf{a}} = \bigoplus_{\mathbf{a} \in \mathbb{Z}A} t^{\mathbf{a}} P_{\mathbf{a}}(\theta_1, \ldots, \theta_d)\mathbb{C}[\theta_1, \ldots, \theta_d],$$

where

$$P_{\mathbf{a}}(\theta_1, \ldots, \theta_d) = \prod_{i=1}^{k} \prod_{j=1}^{-h_i(\mathbf{a})} (h_i(\theta_1, \ldots, \theta_d) - j + 1).$$

(2) Put $D_{\mathbf{a}} = t^{\mathbf{a}}P_{\mathbf{a}}$. Then

$$D_{\mathbf{a}'}D_{\mathbf{a}} = D_{\mathbf{a}+\mathbf{a}'}p_{\mathbf{a}',\mathbf{a}},$$

where

$$\begin{aligned} p_{\mathbf{a}',\mathbf{a}} &= \prod_{h_i(\mathbf{a})>0,\, h_i(\mathbf{a}')<0} \prod_{m=-h_i(\mathbf{a})}^{\min\{-h_i(\mathbf{a}+\mathbf{a}'),0\}-1} (h_i - m) \\ &\times \prod_{h_i(\mathbf{a})<0,\, h_i(\mathbf{a}')>0} \prod_{m=\max\{-h_i(\mathbf{a}+\mathbf{a}'),0\}}^{-h_i(\mathbf{a})-1} (h_i - m). \end{aligned}$$

REMARK 3.2.3. Note that $p_{\mathbf{a}',\mathbf{a}} = q_{-\mathbf{a},-\mathbf{a}'}$ in the notation of [**14**].

PROOF. For the first part, we only need to show that $P_{\mathbf{a}}$ defines the ideal of $\mathbb{C}[\theta_1,\ldots,\theta_d]$ vanishing on $\Omega(\mathbf{a})$. To see this, note that $\mathbf{b} \in \mathbb{N}A$ is in $\Omega(\mathbf{a})$ if and only if $\mathbf{a}+\mathbf{b} \notin \mathbb{N}A$. Because $\mathbb{N}A$ is a saturated cone, this is equivalent to the existence of a boundary functional h_i such that $h_i(\mathbf{a}+\mathbf{b}) < 0$. Since h_i is linear, this just means

$$0 \le h_i(\mathbf{b}) < -h_i(\mathbf{a}).$$

So

$$\mathbb{I}(\Omega(\mathbf{a})) = \prod_{i=1}^{k} \prod_{j=1}^{-h_i(\mathbf{a})} (h_i(\theta_1,\ldots,\theta_d) - j + 1)\mathbb{C}[\theta_1,\ldots,\theta_d].$$

Next we prove the second statement. Since $D_{\mathbf{a}'}D_{\mathbf{a}} \in D(R_A)_{\mathbf{a}+\mathbf{a}'}$, there exists a polynomial $q \in \mathbb{C}[\theta]$ such that $D_{\mathbf{a}'}D_{\mathbf{a}} = D_{\mathbf{a}+\mathbf{a}'}q(\theta)$. Then

$$\begin{aligned} t^{\mathbf{a}+\mathbf{a}'}P_{\mathbf{a}+\mathbf{a}'}(\theta)q(\theta) &= D_{\mathbf{a}+\mathbf{a}'}q(\theta) \\ &= D_{\mathbf{a}'}D_{\mathbf{a}} \\ &= t^{\mathbf{a}'}P_{\mathbf{a}'}(\theta)t^{\mathbf{a}}P_{\mathbf{a}}(\theta) \\ &= t^{\mathbf{a}+\mathbf{a}'}P_{\mathbf{a}'}(\theta+\mathbf{a})P_{\mathbf{a}}(\theta). \end{aligned}$$

Hence $q(\theta) = P_{\mathbf{a}}(\theta)P_{\mathbf{a}'}(\theta+\mathbf{a})/P_{\mathbf{a}+\mathbf{a}'}(\theta) = p_{\mathbf{a}',\mathbf{a}}(\theta)$. □

We say that $\mathbf{a},\mathbf{a}' \in \mathbb{Z}A$ belong to the same chamber if $h_i(\mathbf{a})h_i(\mathbf{a}') \ge 0$ for all i. The following is immediate from Theorem 3.2.2.

COROLLARY 3.2.4. *Define $D_{\mathbf{a}} \in D(\mathbb{C}[\mathbb{Z}^d])$ as in Theorem 3.2.2 regardless of the normality of R_A, i.e.,*

$$D_{\mathbf{a}} = t^{\mathbf{a}} \prod_{i=1}^{k} \prod_{j=1}^{-h_i(\mathbf{a})} (h_i(\theta_1,\ldots,\theta_d) - j + 1).$$

Then $D_{\mathbf{a}'}D_{\mathbf{a}} = D_{\mathbf{a}}D_{\mathbf{a}'}$ if and only if $D_{\mathbf{a}'}D_{\mathbf{a}} = D_{\mathbf{a}+\mathbf{a}'}$ if and only if $\mathbf{a}$ and $\mathbf{a}'$ belong to the same chamber.

COROLLARY 3.2.5. *When R_A is normal, the generators of $D(R_A)$ are the elements of $\{D_{\mathbf{a}}\}_{\mathbf{a}\in\Lambda}$ where Λ contains all generators of the chambers of $\mathbb{N}A$ as well as the origin $\mathbf{0}$. In fact, if we consider the filtration of $D(R_A)$ by order and pass to the graded algebra $Gr(D(R_A))$ then the symbols of these operators generate $Gr(D(R_A))$; that is, $\{D_{\mathbf{a}}\}_{\mathbf{a}\in\Lambda}$ is a canonical subalgebra (SAGBI) basis for $D(R_A)$.*

EXAMPLE 3.2.6. We illustrate the use of Corollary 3.2.5 by computing the generators for $Gr(D(R_A))$ where R_A is the coordinate ring of the twisted cubic. In this case the matrix A is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}.$$

We introduce a convenient abuse of notation: usually one would reserve the term *cone* for $\sigma = \mathbb{R}_{\geq 0}A$; we will also call $\mathbb{N}A$ a cone when the semigroup is saturated. Further, if F is a facet of $\mathbb{R}_{\geq 0}A$, we will refer to $F \cap \mathbb{N}A$ as a facet of $\mathbb{N}A$.

In our example, $\mathbb{N}A$ is a cone (saturated semigroup) bounded by the hyperplanes $h_1 = \theta_2 = 0$ and $h_2 = 3\theta_1 - \theta_2 = 0$. The chambers of $\mathbb{N}A$ are the cones of the fan obtained by extending these two hyperplanes: see Figure 1.

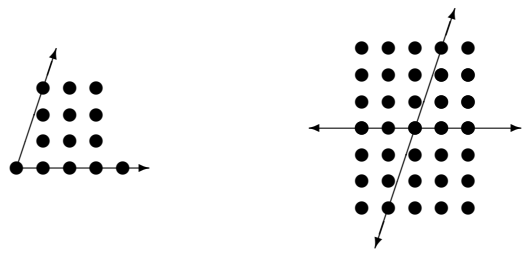


FIGURE 1. The semigroup $\mathbb{N}A$ (left) and a fan illustrating the four chambers (right).

The generators of the four chambers, together with the operators $D_{\mathbf{a}}$ are listed below. For instance, the vector $\mathbf{a} = {}^t(0,1)$ is required to generate the chamber in the second quadrant. The graded piece of $D(R_A)$ of weight $\mathbf{a}$ is a principal module over $D(R_A)_{\mathbf{0}}$ generated by $D_{\mathbf{a}} = t_2 h_2(\theta_1, \theta_2) = 3t_2\theta_1 - t_2\theta_2$. The operators in the right column of the table in Figure 2, together with θ_1 and θ_2 generate $D(R_A)$ as a $\mathbb{C}$-algebra.

Multidegree, $\mathbf{a}$	$D_{\mathbf{a}}$
$(1,0)$	t_1
$(1,1)$	t_1t_2
$(1,2)$	$t_1t_2^2$
$(1,3)$	$t_1t_2^3$
$(0,1)$	t_2h_2
$(-1,0)$	$h_2(h_2-1)(h_2-2)/t_1$
$(-1,-1)$	$h_2(h_2-1)(h_1)/t_1t_2$
$(-1,-2)$	$h_2h_1(h_1-1)/t_1t_2^2$
$(-1,-3)$	$h_1(h_1-1)(h_1-2)/t_1t_2^3$
$(0,-1)$	h_1/t_2

FIGURE 2. Generators of $D(R_A)$ over $\mathbb{C}[\theta_1, \theta_2]$

When R_A is not normal, the ideal $\mathbb{I}(\Omega(\mathbf{a}))$ corresponding to the graded piece $D(R_A)_{\mathbf{a}}$ is not necessarily principal. In the examples below we show that $D(R_A)$ has good properties when R_A is normal but these are apt to fail in the non-normal case.

EXAMPLE 3.2.7. We give a basic example illustrating what can go wrong when R_A is not normal. Consider the affine toric variety $\mathbb{V}(x_1^2 - x_2^2x_3, x_1^3 - x_2^3x_4) \subset \mathbb{C}^4$

associated to the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 1 & 0 & 0 \end{bmatrix};$$

here the coordinate ring is $R = R_A = \mathbb{C}[t_1t_2, t_2, t_1^2, t_1^3]$. The semigroup $\mathbb{N}A$ is illustrated in Figure 3.

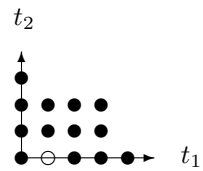


FIGURE 3. The semigroup $\mathbb{N}A$

The ring of differential operators $D(R)$ is finitely generated. To see this, first note that there are four chambers, corresponding to the four quadrants. We claim that the generators of $D(R)$ are:

Multidegree, $\mathbf{a}$	Generators of $\mathbb{I}(\Omega(\mathbf{a}))$
$(1,0)$	θ_1, θ_2
$(2,0)$	1
$(3,0)$	1
$(0,1)$	1
$(-1,0)$	$\theta_1(\theta_1-2)$, $\theta_1\theta_2$
$(-2,0)$	$\theta_1(\theta_1-1)\theta_2$, $\theta_1(\theta_1-1)(\theta_1-3)$
$(0,-1)$	$(\theta_1-1)\theta_2$, $\theta_2(\theta_2-1)$
$(0,-2)$	$(\theta_1-1)\theta_2(\theta_2-1)$, $\theta_2(\theta_2-1)(\theta_2-2)$

The details of this calculation are postponed to Example 3.3.4.

While $D(R)$ is finitely generated, the graded ring $GrD(R)$ is **not** finitely generated. To see this we first construct a projection map from $Gr(D(\mathbb{Z}A)) = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \xi_1, \xi_2]$ to $\mathbb{C}[t_2, \xi_2]$ (here, ξ_1 and ξ_2 stand for the images of ∂_1 and ∂_2 in the graded ring). This induces a ring homomorphism $Gr(D(R)) \hookrightarrow GrD(\mathbb{Z}A) \to \mathbb{C}[t_2, \xi_2]$ whose image is easily checked to be $\mathbb{C}[t_2, t_2\xi_2, t_2\xi_2^2, t_2\xi_2^3, \ldots]$. As the image is not a finitely-generated algebra, the domain $GrD(R)$ cannot be finitely generated.

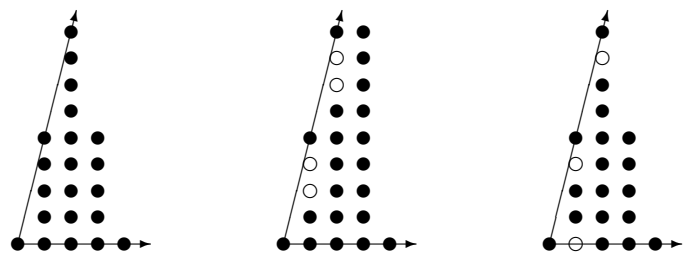
EXAMPLE 3.2.8. We consider the case of the cuspidal cubic: $x_1^2 - x_2^3 = 0$. This has coordinate ring $\mathbb{C}[x_1, x_2]/(x_1^2 - x_2^3) \cong \mathbb{C}[t^2, t^3] = R_A$, where $A = [3\ \ 2]$. There are two chambers for $\mathbb{N}A$ and one can check that $GrD(R)$ is generated by the operators corresponding to weights $\{d : -3 \leq d \leq 3\}$. Direct computation now shows that $GrD(R_A) = \mathbb{C}[t^2, t^3, t\xi, t^2\xi, \xi^2, \xi^3, t\xi^2] = R_B$, where

$$B = \begin{bmatrix} 2 & 3 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 & 3 & 2 \end{bmatrix}.$$

In general, if $GrD(R_A)$ is finitely generated, the Krull dimension of $GrD(R_A)$ is twice that of R_A (see, for example, [**11**] or [**10**, Chapter 15]). This particular example is somewhat unusual in the sense that both R_A and $GrD(R_A)$ are monomial algebras. This occurs whenever R_A has dimension 1 but $GrD(R_A)$ need not be generated by monomials when $dim(R_A) > 1$. For example, consider the semigroup generated by ${}^t(1,0)$, ${}^t(1,1)$ and ${}^t(1,2)$. For many more interesting facts about differential operators on monomial curves (semigroup algebras of dimension one), see [**3**]. These computations suggest the following natural project:

PROBLEM 3.2.9. When $GrD(R_A)$ is finitely generated, describe the structure of the *characteristic variety* $ch(R_A) = Spec(GrD(R_A))$.

We have seen that when $\mathbb{N}A$ is not saturated $GrD(R_A)$ may or may not be finitely generated. Call a semigroup $\mathbb{N}A$ **scored** if its saturation σ satisfies: (1) $\mathbb{R}_{\geq 0}\sigma = \mathbb{R}_{\geq 0}A$ is a strongly convex cone and (2) $\sigma \setminus \mathbb{N}A$ consists of hyperplane sections of σ parallel to facets of σ. Figure 4 illustrates three semigroups. The saturated semigroup in (a) is scored as is the semigroup in (b), obtained by deleting the two hyperplanes $-\theta_2 + 4\theta_1 = 1$ and $-\theta_2 + 4\theta_1 = 2$. Because of the isolated point at (1,0), the semigroup in (c) is not scored.



(a) (b) (c)

FIGURE 4. Scored and nonscored semigroups.

Examples, like those above, suggest the following conjectures.

CONJECTURE 3.2.10. *(1) The graded ring of differential operators $GrD(R_A)$ is finitely generated if and only if $\mathbb{N}A$ is a scored semigroup. (2) The ring of differential operators $D(R_A)$ is finitely generated for all semigroups $\mathbb{N}A$.*

Of course, part (2) of this conjecture is just the content of Corollary 3.2.5 when $\mathbb{N}A$ is saturated. We turn our attention to the necessity of the condition in part (1) of the conjecture. First we state a technical lemma of a combinatorial nature that will be used in the proof of the subsequent Theorem.

LEMMA 3.2.11. *Let $\{v_1, \ldots, v_K\}$ be a set of K vectors all of whose components have absolute value less than $T > 0$. That is, we start with a set of vectors in the ball $\{v \in \mathbb{R}^d : |v|_\infty < T\}$. Suppose that a positive integral combination $\sum_{i=1}^K n_i v_i$ is componentwise greater than or equal to $(3TK, \ldots, 3TK)$. Then all components of $\sum_{i=1}^K \lceil n_i/2 \rceil v_i$ and $\sum_{i=1}^K \lfloor n_i/2 \rfloor v_i$ are greater than or equal to KT.*

PROOF. Note

$$\left| \sum_{i=1}^K \lceil n_i/2 \rceil v_i - \sum_{i=1}^K (n_i/2) v_i \right|_\infty \leq KT/2.$$

Thus, each component of $\sum_{i=1}^K \lceil n_i/2 \rceil v_i$ is larger than the corresponding component of $\sum_{i=1}^K (n_i/2) v_i$ minus $KT/2$. But this is at least $\frac{3KT}{2} - \frac{KT}{2} = KT$. The claim for $\sum_{i=1}^K \lfloor n_i/2 \rfloor v_i$ is proven similarly. □

THEOREM 3.2.12. *When $\mathbb{N}A$ is a semigroup that is not scored then $GrD(R_A)$ is not finitely generated.*

PROOF. We aim for a contradiction and suppose that $GrD(R_A)$ is finitely generated and $\mathbb{N}A$ is not a scored semigroup. Let σ be the saturation of $\mathbb{N}A$. Suppose that the symbols of the K operators $t^{\mathbf{g}_i}Q_i(\theta)$ $(i = 1, \ldots, K)$ generate $GrD(R_A)$ and pick $T > 0$ such that for all facets F of σ and for all generators, $|h_F(-\mathbf{g_i})| < T$. (This just says that the generators are not too far from the origin).

Note that $\mathbb{N}A$ contains a translate of its saturation σ: there exists an $\alpha \in \mathbb{N}A$ such that $\alpha + \sigma \subset \mathbb{N}A$. The reader can examine [**5**] for results of this kind but this particular result follows immediately from the fact that the conductor of $\mathbb{C}[\sigma]$ into $\mathbb{C}[\mathbb{N}A]$ is graded and nonempty. Modifying T if necessary, we may assume that $h_F(\alpha) < T$ for all facets F of σ. It follows that if $\beta \in \mathbb{N}A$ satisfies $h_F(\beta) \geq T$ for all facets F of σ, then $\beta \in \alpha + \sigma$.

Let L be the smallest non-negative integer such that if $\gamma \in \alpha+\sigma$, then all generators of $B_\gamma := \mathbb{I}(\Omega(-\gamma))$ have degree larger or equal to $L+\sum_F h_F(\gamma)$. The number L is strictly positive because of our assumption on $\mathbb{N}A$ (the semigroup is not scored and so $\Omega(-\gamma)$ consists of more than just the expected number $\sum_F max\{h_F(\gamma), 0\}$ of hyperplanes parallel to facets).

Recall that we are assuming that there are K generators for $GrD(R_A)$; now pick $\gamma \in \alpha + \sigma$ with $h_F(\gamma) > KT$ for all facets F. We know that there is an operator $P = t^{-3\gamma}Q$ such that the degree of Q equals $L + \sum_F h_F(3\gamma)$. Write the symbol of P as a sum of products $\prod_{k=1}^s P_k$ where P_k is the symbol of $t^{-\mathbf{g}_{i_k}}Q_{i_k}$ $(Q_{i_k} \in \mathbb{I}(\Omega(-\mathbf{g}_{i_k})))$. Now using Lemma 3.2.11 applied to the vectors $(h_F(\mathbf{g}_{i_k}) :$ F a facet of $\sigma)$, the index set $\{1, 2, \ldots, s\}$ can be partitioned into two disjoint sets R and S such that for each facet F, $h_F(\sum_{k\in R} \mathbf{g}_{i_k}) \geq T$ and $h_F(\sum_{k\in S} \mathbf{g}_{i_k}) \geq T$. That is, $\sum_{k\in R} \mathbf{g}_{i_k} \in \alpha+\sigma$ and $\sum_{k\in S} \mathbf{g}_{i_k} \in \alpha+\sigma$. It follows that the degree of $\prod_{k\in R} Q_{i_k}$ is greater than or equal to $L + \sum_F h_F(\sum_{k\in R} g_{i_k})$ and the degree of $\prod_{k\in S} Q_{i_k}$ is greater than or equal to $L + \sum_F h_F(\sum_{k\in S} g_{i_k})$. But then

$$\begin{array}{rcl}
\deg(\prod_{k=1}^s Q_{i_k}) & = & \deg(\prod_{k\in R} Q_{i_k}) + \deg(\prod_{k\in S} Q_{i_k}) \\
& \geq & (L + \sum_F h_F(\sum_{k\in R} g_{i_k})) + (L + \sum_F h_F(\sum_{k\in S} g_{i_k})) \\
& = & 2L + \sum_F h_F(\sum_{k=1}^s g_{i_k}) \\
& = & 2L + \sum_F h_F(3\gamma) \\
& = & L \ + \ \deg(Q) \\
& > & \deg(Q).
\end{array}$$

It follows that the symbol of P (that is, $t^{-3\gamma}\text{symbol}(Q)$) cannot be generated by the symbols of elements of weights $\{\mathbf{g_i}\}$. So $GrD(R_A)$ is not finitely generated. □

It seems difficult to establish the first part of Conjecture 3.2.10 using computational methods. However, we can do this in the special case that the cone $\mathbb{R}_{\geq 0}A$ is generated by linearly independent generators.

THEOREM 3.2.13. *When $\mathbb{N}A$ is a scored semigroup and $\mathbb{R}_{\geq 0}A$ is generated by linearly independent vectors, then $GrD(R_A)$ is finitely generated.*

PROOF. Let $\{u_1, \ldots, u_d\} \subset \mathbb{N}A$ be a linearly independent set of elements generating $\mathbb{R}_{\geq 0}A$. For each facet F of the cone $\mathbb{R}_{\geq 0}A$, there is a unique generator $u_F \in \{u_1, \ldots, u_d\}$ such that $u_F \notin F$. Let d_F be the maximum of $h_F(u_F)$ and the number of hyperplanes parallel to F missing from $\mathbb{N}A$. Our generators will be the symbols $P_\mathbf{a}$ of those operators $Q_\mathbf{a}$ with $-d_F \leq h_F(\mathbf{a}) \leq 2d_F$ for each facet F. Call the set of such generators $G = \{P_a\}_{\mathbf{a}\in\Lambda}$. Both u_F and $-u_F$ are in the index set Λ for each facet F.

Note that if $\mathbf{v} \in \mathbb{Z}A$ has $h_F(\mathbf{v}) \geq d_F$ then $\Omega(-\mathbf{v})$ consists of $h_F(\mathbf{v})$ hyperplanes parallel to F (plus hyperplanes parallel to other facets). Because $\Omega(-\mathbf{v})$ consists only of hyperplanes, $GrD(R)_{-\mathbf{v}}$ is principally generated and has generator divisible by $h_F^{h_F(\mathbf{v})}$. Similarly, if $h_F(\mathbf{v}) < -d_F$ then $\Omega(-\mathbf{v})$ consists of hyperplanes parallel to other facets of $\mathbb{N}A$, but no hyperplanes parallel to F. In this case, the generator of $GrD(R)_{-\mathbf{v}}$ is not divisible by h_F.

Now we show that the operators with weights in Λ generate $GrD(R)$. For $\mathbf{b} \in \mathbb{Z}A$ with $h_F(\mathbf{b}) \geq 2d_F$ there exists $\mathbf{a} \in \Lambda$ and a positive integer r such that $\mathbf{a} + r \cdot u_F = \mathbf{b}$. Moreover, $GrD(R)_{-\mathbf{b}}$ is generated by $P_{-\mathbf{b}} = P_{-\mathbf{a}} t^{-ru_F} (h_F)^{r \cdot h_F(u_F)} = P_{-\mathbf{a}} P^r_{-u_F}$. Similarly, for $\mathbf{b} \in \mathbb{Z}A$ with $h_F(\mathbf{b}) < -d_F$, there exists $\mathbf{a} \in \Lambda$ and a positive integer r such that $\mathbf{a} - r \cdot u_F = \mathbf{b}$. Then $GrD(R)_{-\mathbf{b}}$ is generated by $P_{-\mathbf{b}} = P_{-\mathbf{a}} P^r_{u_F}$. Iterating this procedure for various facets F, we can write any given weight vector $\mathbf{b}$ as an integral linear combination of the u_F's and another element $\mathbf{a} \in \Lambda$. Then $P_{-\mathbf{b}}$ is generated by $P_{\pm u_F}$ and $P_{-\mathbf{a}}$. This shows that the set G generates $GrD(R)$.

□

Remark 3.2.14. Together, Theorems 3.2.13 and 3.2.12 imply Conjecture 3.2.10 in dimensions one and two.

Theorem 3.2.13 generalizes a result of Eriksen and Vosegaard [**3**] in the one-dimensional case: if the last hole in a one-dimensional semigroup appears at position g then it suffices to use generators corresponding to weights γ with $-g \leq \gamma \leq 2g+1$ in order to generate $GrD(R_A)$.

The ring of differential operators can be viewed as a special instance of the module of differential operators $D(M, N)$ from one R-module, M, to another, N. For details on this construction, see [**18**]. Part of our interest in the finite generation of $GrD(R)$ stems from a conjecture due to Joseph Becker. Becker's conjecture deals with the module of constant coefficient differential operators $D(R, R/m)$ (here R is assumed to be local or graded with maximal (homogeneous) ideal m). Define $D(R, R/m)$ as

$$\frac{\{\theta \in W : \ \theta * I \subseteq m\}}{mW}.$$

In the graded case, it can be shown that $D(R, R/m)$ is precisely the graded dual of R, that is, $D(R, R/m) = Hom^*_k(R, k)$, the injective hull of the residue field of R (see [**23**]). Note that there is a map of R-modules $\gamma : D(R) \to D(R, R/m)$ induced by the quotient map $\rho : R \to R/m$. It may help to think of $\gamma(\delta) = \rho \circ \delta$ as the constant coefficient portion of the differential operator δ.

Conjecture 3.2.15 (Becker's Conjecture). *If $GrD(R)$ is finitely generated as an R-algebra then the map $D(R) \to D(R, R/m)$ from differential operators to constant coefficient differential operators is surjective.*

This conjecture is known to imply Nakai's conjecture: when R is a finitely generated $\mathbb{C}$-algebra and $D(R)$ is generated as an R-algebra by derivations then R is smooth over $\mathbb{C}$ (see [**1**]).

We will verify Becker's Conjecture for semigroup algebras. We begin with an explicit description of the module of constant coefficient differential operators on R_A that was suggested to us by Bernd Sturmfels. First note that the module of constant coefficient differential operators is graded: $D(R_A, R_A/m_A) =$

$\oplus_{\mathbf{a}} D(R_A, R_A/m_A)_{\mathbf{a}\in -\mathbb{N}A}$. The graded piece of weight $\mathbf{a}$ is a 1-dimensional $\mathbb{C}$-vector space generated by the linear map that sends $x^{\mathbf{a}}$ to 1 and kills all other monomials in R_A. Presenting the algebra R_A,

$$0 \to I_A \to \mathbb{C}[x_1, \ldots, x_n] \to R_A \to 0,$$

this map can be expressed as the differential operator

$$T_{\mathbf{a}} = \sum_{\{\mathbf{b}\in\mathbb{N}^n:\, A\mathbf{b}=\mathbf{a}\}} \prod_{i=1}^{n} \frac{1}{b_i!} \frac{\partial^{b_i}}{\partial x_i^{b_i}};$$

here the strong convexity of $\mathbb{N}A$ guarantees that the sum is finite. To check this formula defines the desired map, note that $T_{\mathbf{a}}$ sends I_A to m_A and sends any representative $x^{\mathbf{b}}$ of $t^{\mathbf{a}}$ to 1. Furthermore, by examining the grading, $T_{\mathbf{a}}$ sends all other monomials in R_A to an element of m_A. We have established the following result:

Lemma 3.2.16. *If R_A is a semigroup algebra with homogeneous maximal ideal m_A, then*

$$D(R_A, R_A/m_A) = \oplus_{\mathbf{a}\in\mathbb{N}A}\, \mathbb{C} \cdot \left(\sum_{\{\mathbf{b}:\, A\mathbf{b}=\mathbf{a}\}} \prod_{i=1}^{n} \frac{1}{b_i!} \frac{\partial^{b_i}}{\partial x_i^{b_i}} \right).$$

Theorem 3.2.17. *Becker's Conjecture is true for semigroup algebras.*

Proof. Consider a semigroup algebra R_A with $GrD(R_A)$ finitely generated. By Theorem 3.2.12 $\mathbb{N}A$ is a scored semigroup. Given $\mathbf{a} \in \mathbb{N}A$, we argue from the structure theorem that any generator $P_{\mathbf{a}}$ of $D(R_A)_{-\mathbf{a}}$ maps to a constant multiple of $T_{\mathbf{a}}$ under the map $\gamma : D(R_A) \to D(R_A, R_A/m_A)$. By considering the grading, we see that $P_{\mathbf{a}}$ sends all monomials save possibly $t^{\mathbf{a}}$ into the ideal m_A. Furthermore, $t^{\mathbf{a}}$ is sent to a multiple of $1 = t^{\mathbf{0}}$. Since $\mathbb{N}A$ is scored, the set $\Omega(-\mathbf{a})$ is a union of hyperplanes parallel to the facets, none of which pass through $\mathbf{a}$. It follows that none of the minimal generators of $D(R_A)_{-\mathbf{a}}$ kill $t^{\mathbf{a}}$. From this it follows that $\gamma(D(R_A)_{-\mathbf{a}}) = \mathbb{C}T_{\mathbf{a}}$ and hence by Lemma 3.2.16 the map $\gamma : D(R_A) \to D(R_A, R_A/m_A)$ is surjective. □

Remark 3.2.18. The proof above uses Theorem 3.2.12 but this hides the key idea: when $GrD(R_A)$ is finitely generated, the algebra R_A is simple as a $D(R_A)$-algebra. Indeed, as Karen E. Smith pointed out [**17**], the surjectivity of the map $D(R) \to D(R, R/m)$ is equivalent to the D-simplicity of R in a much more general setting. We give a characterization of D-simple semigroup algebras in Theorem 4.1.6.

By establishing Becker's conjecture for semigroup algebras we give a new proof that Nakai's conjecture holds for semigroup algebras. For another proof, based on the fact that the normalization of any semigroup algebra is D-simple, see [**22**].

3.3. Algorithmic Description. In the theory of the symmetry algebra, the ideal of *b-polynomials* plays an important role (see [**15**]). From our viewpoint, the ideal of b-polynomials $B_{-\mathbf{a}}$ is nothing but $t^{-\mathbf{a}} D(R_A)_{\mathbf{a}}$. In this subsection, we review how to compute $B_{-\mathbf{a}}$. But first let us restate the structure theorem for $D(R_A)$, discussed in the previous subsection.

THEOREM 3.3.1.

$$D(R_A) = \bigoplus_{\mathbf{a}\in\mathbb{Z}A} D(R_A)_{\mathbf{a}} = \bigoplus_{\mathbf{a}\in\mathbb{Z}A} t^{\mathbf{a}} B_{-\mathbf{a}},$$

where

$$B_{-\mathbf{a}} = t^{-\mathbf{a}} D(R_A)_{\mathbf{a}} = \mathbb{I}(\Omega(\mathbf{a})).$$

The ideal $B_{-\mathbf{a}}$ is related to the standard pairs of a certain monomial ideal $M_{-\mathbf{a}} \subset \mathbb{C}[\mathbf{x}]$. Let

$$\begin{aligned} M_{-\mathbf{a}} : &= \langle\, x^{\mathbf{w}} \mid A\mathbf{w} \in -\mathbf{a} + \mathbb{N}A \,\rangle \\ &= \mathrm{mono}(\langle\, x^{\mathbf{u}} \,\rangle + I_A : x^{\mathbf{v}}), \end{aligned}$$

where $\mathbf{a} = A\mathbf{v} - A\mathbf{u}$ ($\mathbf{u}, \mathbf{v} \in \mathbb{N}^n$), and mono stands for the operation of taking the largest monomial subideal. Algorithm 4.4.2 in [**16**] gives a procedure to compute the largest monomial subideal.

For each monomial ideal $M \subset C[\mathbf{x}]$ we can decompose the set of monomials not in M into *standard pairs*. A pair $(\mathbf{u}, \tau)$ with $\mathbf{u} \in \mathbb{N}^n$ and $\tau \subset \{1, \ldots, n\}$ is called a *standard pair* of M if it satisfies the following conditions:

(1) $u_j = 0$ for all $j \in \tau$. (We abbreviate this to $\mathbf{u} \in \mathbb{N}^{\tau^c}$, where ^{c} stands for the operation of taking the complement.)
(2) There exists no $\mathbf{v} \in \mathbb{N}^\tau$ such that $x^{\mathbf{u}+\mathbf{v}} \in M$.
(3) For each $j \notin \tau$, there exists $\mathbf{v} \in \mathbb{N}^{\tau\cup\{j\}}$ such that $x^{\mathbf{u}+\mathbf{v}} \in M$.

See ([**6**], [**16**, Algorithm 3.2.5]) for algorithms to compute the standard pair decomposition of a monomial ideal.

LEMMA 3.3.2 (Lemma 4.4 in [**15**]). *Let $(\mathbf{u}, \tau)$ be a standard pair of $M_{-\mathbf{a}}$. Then $\sum_{j\in\tau} \mathbb{R}_{\geq 0}\mathbf{a}_j$ is a proper face of $\mathbb{R}_{\geq 0}A$, and moreover $\tau = \{\, i \mid \mathbf{a}_i \in \sum_{j\in\tau} \mathbb{R}_{\geq 0}\mathbf{a}_j \,\}$.*

Thanks to Lemma 3.3.2, we may identify τ appearing in a standard pair, with a proper face, also denoted τ, of $\mathbb{R}_{\geq 0}A$.

Let $\mathcal{S}(M_{-\mathbf{a}})$ be the set of standard pairs of $M_{-\mathbf{a}}$. As usual, for an ideal I of $\mathbb{C}[\theta] = \mathbb{C}[\theta_1, \ldots, \theta_d]$, we denote by $\mathbb{V}(I)$ the zero set of I.

THEOREM 3.3.3 (Theorem 4.5 in [**15**]). *Let σ_i be the facet of the cone $\mathbb{R}_{\geq 0}A$ corresponding to h_i. Then we have the following.*

(1)

$$\Omega(\mathbf{a}) = \{\, A\mathbf{u} + \mathbb{N}(A \cap \tau) \mid (\mathbf{u}, \tau) \in \mathcal{S}(M_{-\mathbf{a}}) \,\}.$$

(2)

$$B_{-\mathbf{a}} = \mathbb{I}(\Omega(\mathbf{a})) = \bigcap_{(\mathbf{u},\tau)\in\mathcal{S}(M_{-\mathbf{a}})} \langle\, h_i - h_i(A\mathbf{u}) \mid \sigma_i \text{ :facet} \supset \tau \rangle.$$

(3)

$$\mathbb{V}(B_{-\mathbf{a}}) = \bigcup_{(\mathbf{u},\tau)\in\mathcal{S}(M_{-\mathbf{a}})} (A\mathbf{u} + \mathbb{C}(A \cap \tau)).$$

We can use Theorem 3.3.3 to compute $\mathbb{I}(\Omega(\mathbf{a}))$.

EXAMPLE 3.3.4 (Continuation of Example 3.2.7). From Lemma 3.3.2, each standard pair of $M_{-\mathbf{a}}$ is of one of the forms: $(u_1, u_2, *, *)$, $(u_1, *, u_3, u_4)$, (u_1, u_2, u_3, u_4).

Here we use the shorthand $(u_1, u_2, *, *)$ to denote the standard pair $((u_1, u_2, 0, 0), 3, 4)$; in general, τ is the set of components containing $*$'s. Now

$$\begin{aligned}\mathcal{S}(M_{-\mathbf{a}}) &= \{\,(u_1, u_2, *, *) \mid u_1 + u_2 < -a_2\,\} \\ &\quad \cup \{\,(u_1, *, u_3, u_4) \mid u_1 + 2u_3 + 3u_4 < -a_1\,\} \\ &\quad \cup \{\,(u_1, u_2, u_3, u_4) \mid u_1 + 2u_3 + 3u_4 = -a_1 + 1,\ u_1 + u_2 = -a_2\,\}.\end{aligned}$$

$$\mathbb{I}(\Omega(\mathbf{a})) = \langle\, P_{\mathbf{a}} \,\rangle \qquad \text{if } \mathbf{a} \notin {}^t(1,0) - \mathbb{N}A,$$

$$\mathbb{I}(\Omega(\mathbf{a})) = \langle\, P_{\mathbf{a}} \,\rangle \times \langle\, \theta_1 + a_1 - 1, \theta_2 + a_2 \,\rangle \qquad \text{if } \mathbf{a} \in {}^t(1,0) - \mathbb{N}A,$$

where $P_{\mathbf{a}} = \prod_{m=0}^{-a_1-1}(\theta_1 - m) \cdot \prod_{m=0}^{-a_2-1}(\theta_2 - m)$ as in Theorem 3.2.2.

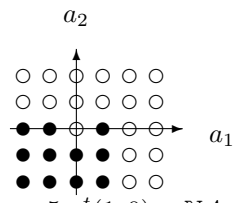


FIGURE 5. ${}^t(1,0) - \mathbb{N}A$

Put $D_{\mathbf{a}} := t^{\mathbf{a}} P_{\mathbf{a}} \in D(\mathbb{C}[\mathbb{Z}^d])$, $E_{\mathbf{a}} := D_{\mathbf{a}}(\theta_1 + a_1 - 1)$, and $F_{\mathbf{a}} := D_{\mathbf{a}}(\theta_2 + a_2)$. Then

$$\begin{aligned} D(R_A)_{\mathbf{a}} &= D_{\mathbf{a}}\mathbb{C}[\theta] && \text{if } \mathbf{a} \notin {}^t(1,0) - \mathbb{N}A, \\ D(R_A)_{\mathbf{a}} &= E_{\mathbf{a}}\mathbb{C}[\theta] + F_{\mathbf{a}}\mathbb{C}[\theta] && \text{if } \mathbf{a} \in {}^t(1,0) - \mathbb{N}A. \end{aligned}$$

If $\mathbf{a}$ and $\mathbf{a}'$ belong to the same chamber (quadrant), then we have the following formulas (cf. Corollary 3.2.4):

$$\begin{aligned} &D_{\mathbf{a}} D_{\mathbf{a}'} = D_{\mathbf{a}+\mathbf{a}'} = D_{\mathbf{a}'} D_{\mathbf{a}} \\ &[D_{\mathbf{a}}, E_{\mathbf{a}'}] = -a_1 D_{\mathbf{a}+\mathbf{a}'} \\ &[D_{\mathbf{a}}, F_{\mathbf{a}'}] = -a_2 D_{\mathbf{a}+\mathbf{a}'} \\ &[E_{\mathbf{a}}, E_{\mathbf{a}'}] = (a_1' - a_1) E_{\mathbf{a}+\mathbf{a}'} \\ &[F_{\mathbf{a}}, F_{\mathbf{a}'}] = (a_2' - a_2) F_{\mathbf{a}+\mathbf{a}'} \\ &[E_{\mathbf{a}}, F_{\mathbf{a}'}] = D_{\mathbf{a}+\mathbf{a}'}(a_1'(\theta_2 + a_2') - a_2(\theta_1 + a_1 - 1)). \end{aligned}$$

The above formulas imply that the algebra $D(R_A)$ is generated by $D_{{}^t(0,1)}$, $D_{{}^t(2,0)}$, $D_{{}^t(3,0)}$, $E_{{}^t(1,0)}$, $F_{{}^t(1,0)}$, $E_{{}^t(0,-1)}$, $F_{{}^t(0,-1)}$, $E_{{}^t(0,-2)}$, $F_{{}^t(0,-2)}$, $E_{{}^t(-1,0)}$, $F_{{}^t(-1,0)}$, $E_{{}^t(-2,0)}$, $F_{{}^t(-2,0)}$ together with θ_1 and θ_2.

3.4. Application to A-hypergeometric systems. Associated to a face τ of the cone $\sigma = \mathbb{R}_{\geq 0}A$ and a parameter $\mathbf{a} \in \mathbb{C}^d$, we introduced a finite set $E_\tau(\mathbf{a})$ in [**15**]:

$$E_\tau(\mathbf{a}) := \{\,\mathbf{l} \in \mathbb{C}(A \cap \tau)/\mathbb{Z}(A \cap \tau) \mid \mathbf{a} - \mathbf{l} \in \mathbb{N}A + \mathbb{Z}(A \cap \tau)\,\}.$$

EXAMPLE 3.4.1 (Continuation of Examples 3.2.7 and 3.3.4). Let us consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

again. The cone $\sigma = \mathbb{R}_{\geq 0}A$ has four faces: σ itself, the nonnegative part of the t_1-axis σ_1, the nonnegative part of the t_2-axis σ_2, and the origin $\{\,\mathbf{0}\,\}$. We have

$$\mathbb{Z}(A \cap \tau) = \begin{cases} \mathbb{Z}A & \text{if } \tau = \sigma \\ \mathbb{Z}^t(1,0) & \text{if } \tau = \sigma_1 \\ \mathbb{Z}^t(0,1) & \text{if } \tau = \sigma_2 \\ \{\,0\,\} & \text{if } \tau = \{\,\mathbf{0}\,\}. \end{cases}$$

We shall compute $E_\tau(\mathbf{a})$ for all $\mathbf{a} \in \mathbb{Z}A$ and for all faces τ. Suppose $\mathbf{a} \in \mathbb{Z}A$. Since $\mathbb{N}A + \mathbb{Z}(A \cap \tau)$ is a subset of $\mathbb{Z}A$, $\mathbf{l} \in E_\tau(\mathbf{a})$ implies that $\mathbf{l} \in \mathbb{Z}A$, and thus $\mathbf{l} \in \mathbb{Z}A \cap \mathbb{C}(A \cap \tau)$. In our case, $\mathbb{Z}A \cap \mathbb{C}(A \cap \tau) = \mathbb{Z}(A \cap \tau)$ for all four faces τ. Hence for each τ, there are only two possibilities: $E_\tau(\mathbf{a}) = \{\,0\,\}$ or $E_\tau(\mathbf{a}) = \emptyset$. The condition $E_\sigma(\mathbf{a}) = \{\,0\,\}$ is equivalent to the condition $\mathbf{a} \in \mathbb{Z}A$, which we assume. For the other three faces τ, we have

$$E_{\sigma_1}(\mathbf{a}) = \{\,0\,\} \Leftrightarrow \mathbf{a} \in \mathbb{N}A + \mathbb{Z}^t(1,0) \Leftrightarrow a_2 \geq 0.$$
$$E_{\sigma_2}(\mathbf{a}) = \{\,0\,\} \Leftrightarrow \mathbf{a} \in \mathbb{N}A + \mathbb{Z}^t(0,1) \Leftrightarrow a_1 \geq 0.$$
$$E_{\{\,\mathbf{0}\,\}}(\mathbf{a}) = \{\,0\,\} \Leftrightarrow \mathbf{a} \in \mathbb{N}A.$$

Note that $\mathbf{a} = {}^t(1,0)$ is the unique point with $E_{\sigma_1}(\mathbf{a}) = \{\,0\,\}$, $E_{\sigma_2}(\mathbf{a}) = \{\,0\,\}$, and $E_{\{\,\mathbf{0}\,\}}(\mathbf{a}) = \emptyset$.

The finite sets $E_\tau(\mathbf{a})$ can be computed by the following two algorithms.

Algorithm 3.4.2. *Input:* $\mathbf{a} = A\mathbf{u}_+ - A\mathbf{u}_-$ $(\mathbf{u}_+, \mathbf{u}_- \in \mathbb{N}^n)$.
Output: $E = E_\tau(\mathbf{a})$.

(1) *Take any set E of complete representatives of the set $\mathbb{Q}(A \cap \tau) \cap \mathbb{Z}A / \mathbb{Z}(A \cap \tau)$.*
(2) *For each $\mathbf{l} \in E$, choose any $\mathbf{l}_+, \mathbf{l}_- \in \mathbb{N}^n$ with $\mathbf{l} = A\mathbf{l}_+ - A\mathbf{l}_-$.*
(3) *For $\mathbf{l} \in E$, if $x^{\mathbf{u}_+ + \mathbf{l}_-} \notin (\langle x^{\mathbf{u}_- + \mathbf{l}_+} \rangle + I_A : x_j^\infty(\mathbf{a}_j \in \tau))$, then $E := E \setminus \{\,\mathbf{l}\,\}$.*

Proof. This follows from the fact that $\mathbf{l} \in E_\tau(\mathbf{a})$ if and only if $A(\mathbf{u}_+ + \mathbf{l}_-) \in A(\mathbf{u}_- + \mathbf{l}_+) + \mathbb{N}A + \mathbb{Z}(A \cap \tau)$. □

Algorithm 3.4.3. *Input:* $\mathbf{a} = A\mathbf{u}$ $(\mathbf{u} \in \mathbb{C}^n)$.
Output: $E = E_\tau(\mathbf{a})$.

(1) *Find $\mathbf{m} \in \mathbb{C}(A \cap \tau)$ such that $\mathbf{a} - \mathbf{m} \in \mathbb{Z}A$. If no such $\mathbf{m}$ exists, then $E := \emptyset$ and STOP. Otherwise GO TO Step 2.*
(2) *Compute $E_\tau(\mathbf{a} - \mathbf{m})$ using Algorithm 3.4.2. Put $E := \mathbf{m} + E_\tau(\mathbf{a} - \mathbf{m})$.*

Proof. This is immediate from the equivalence of $\mathbf{l} \in E_\tau(\mathbf{a})$ with $\mathbf{l} - \mathbf{m} \in E_\tau(\mathbf{a} - \mathbf{m})$. □

Recall that the *A-hypergeometric system with parameter* $\mathbf{a}$ is just the W-module

$$M_A(\mathbf{a}) := W/(WI_A(\partial) + \sum_{i=1}^{d} W(\sum_{j=1}^{n} a_{ij}x_j\partial_j - a_i)).$$

Using the finite sets $E_\tau(\mathbf{a})$, the first author classified $M_A(\mathbf{a})$ in the homogeneous case; that is, assuming the columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ of A lie on a hyperplane not passing through the origin (see Theorem 2.1 in [**15**]). We generalize this result to the nonhomogeneous case.

Theorem 3.4.4. *$M_A(\mathbf{a}) \simeq M_A(\mathbf{a}')$ if and only if $E_\tau(\mathbf{a}) = E_\tau(\mathbf{a}')$ for all faces τ.*

Proof. In the proof of Theorem 2.1 in [**15**], the assumption that A is homogeneous is used to derive properties of the symmetry algebra $S(W/WI_A(\partial))$ (Theorem 3.3.1 and Proposition 3.4 in [**15**]). Since analogous results hold in general for $D(R_A)$, the theorem follows from the anti-isomorphism of $D(R_A) \subset D(\mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}])$ and $S(W/WI_A(\partial))$. □

4. $D(R)$-modules

4.1. In this section we determine the $D(R_A)$-module structure of $t^{\mathbf{a}} \cdot R_A^{\pm} = x^{\mathbf{v}} \cdot \mathbb{C}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]/I_A$ for $\mathbf{a} = A\mathbf{v} \in \mathbb{C}^d$.

Definition 4.1.1. We define a partial order $\preceq$ in the parameter space $\mathbb{C}^d = \mathbb{C}A$ as follows: for $\mathbf{a}, \mathbf{b} \in \mathbb{C}^d$ say $\mathbf{a} \preceq \mathbf{b}$ when $E_\tau(\mathbf{a}) \subseteq E_\tau(\mathbf{b})$ for all faces τ of the cone σ. We say that $\mathbf{a}$ and $\mathbf{b}$ are equivalent, denoted by $\mathbf{a} \sim \mathbf{b}$, if $\mathbf{a} \preceq \mathbf{b}$ and $\mathbf{a} \succeq \mathbf{b}$. We use the notation $\mathbf{a} \prec \mathbf{b}$ if $\mathbf{a} \preceq \mathbf{b}$ and $\mathbf{a} \nsim \mathbf{b}$.

Remark 4.1.2. Suppose that $\mathbf{a} \preceq \mathbf{b}$. Then $E_\sigma(\mathbf{a}) \subseteq E_\sigma(\mathbf{b})$. Since both $E_\sigma(\mathbf{a})$ and $E_\sigma(\mathbf{b})$ consist of one element ([**15**, Proposition 2.2.1]), this implies that $E_\sigma(\mathbf{a}) = E_\sigma(\mathbf{b})$; that is, $\mathbf{a} - \mathbf{b} \in \mathbb{Z}A$.

Lemma 4.1.3.

$$R_A \cdot x^{\mathbf{a}} \subseteq \bigoplus_{\mathbf{b} \succeq \mathbf{a}} \mathbb{C}x^{\mathbf{b}}.$$

Proof. This is immediate from the fact that $\mathbf{a} \preceq \mathbf{a} + \mathbf{c}$ for all $\mathbf{c} \in \mathbb{N}A$ ([**15**, Proposition 2.2.5]). □

The following is the key lemma of this section.

Lemma 4.1.4. *$\mathbf{a} \not\preceq \mathbf{b}$ if and only if $\mathbf{a} \in \mathbb{V}(B_{\mathbf{a}-\mathbf{b}})$.*

Proof. First suppose $\mathbf{a} \in \mathbb{V}(B_{\mathbf{a}-\mathbf{b}})$. Then by Theorem 3.3.3, there exists a standard pair $(\mathbf{u}, \tau) \in \mathcal{S}(M_{\mathbf{a}-\mathbf{b}})$ such that $\mathbf{a} - A\mathbf{u} \in \mathbb{C}(A \cap \tau)$. If $E_\tau(\mathbf{a}) \subseteq E_\tau(\mathbf{b})$, then $\mathbf{b} - (\mathbf{a} - A\mathbf{u}) \in \mathbb{N}A + \mathbb{Z}(A \cap \tau)$, which implies $(A\mathbf{u} + \mathbb{N}(A \cap \tau)) \cap ((\mathbf{a} - \mathbf{b}) + \mathbb{N}A) \neq \emptyset$. This contradicts the assumption that $(\mathbf{u}, \tau)$ is a standard pair. Hence $\mathbf{a} \not\preceq \mathbf{b}$.

Next suppose that $\mathbf{a} \not\preceq \mathbf{b}$. Take $\mathbf{l} \in E_\tau(\mathbf{a}) \setminus E_\tau(\mathbf{b})$; that is, $\mathbf{a} - \mathbf{l} \in \mathbb{N}A + \mathbb{Z}(A \cap \tau)$ and $\mathbf{b} - \mathbf{l} \notin \mathbb{N}A + \mathbb{Z}(A \cap \tau)$. Then there exists $\mathbf{u} \in \mathbb{N}^{\tau^c}$ such that $\mathbf{a} - A\mathbf{u} \in \mathbf{l} + \mathbb{Z}(A \cap \tau)$ and $\mathbf{b} - (\mathbf{a} - A\mathbf{u}) \notin \mathbb{N}A + \mathbb{Z}(A \cap \tau)$. The latter statement is equivalent to $(A\mathbf{u} + \mathbb{N}(A \cap \tau)) \cap ((\mathbf{a} - \mathbf{b}) + \mathbb{N}A) = \emptyset$. Hence there exists a standard pair $(\tilde{\mathbf{u}}, \tilde{\tau}) \in \mathcal{S}(M_{\mathbf{a}-\mathbf{b}})$ such that $\tilde{\tau} \succeq \tau$ and $\mathbf{u}_{|\tilde{\tau}^c} = \tilde{\mathbf{u}}$. Then $\mathbf{a} - A\tilde{\mathbf{u}} \in A\mathbf{u}_{|\tau^c \cap \tilde{\tau}} + \mathbf{l} + \mathbb{Z}(A \cap \tau) \subseteq \mathbb{C}(A \cap \tilde{\tau})$. Hence $\mathbf{a} \in \mathbb{V}(B_{\mathbf{a}-\mathbf{b}})$. □

Proposition 4.1.5. (1) *If $\mathbf{a} \preceq \mathbf{b}$, then $x^{\mathbf{b}} \in D(R_A) * x^{\mathbf{a}}$.*
(2) *$\bigoplus_{\mathbf{b} \succeq \mathbf{a}} \mathbb{C}x^{\mathbf{b}}$ is a $D(R_A)$-module.*

Proof. Let $\mathbf{a} - \mathbf{b} \in \mathbb{Z}A$, $p(\theta) \in B_{\mathbf{a}-\mathbf{b}}$, and $P = x^{\mathbf{b}-\mathbf{a}} p(\theta) \in D(R)_{\mathbf{b}-\mathbf{a}}$. Then $P * x^{\mathbf{a}} = p(\mathbf{a}) x^{\mathbf{b}}$.

Suppose that $\mathbf{a} \preceq \mathbf{b}$. Then by Lemma 4.1.4, there exists a polynomial $p(\theta) \in B_{\mathbf{a}-\mathbf{b}}$ such that $p(\mathbf{a}) \neq 0$. With this choice of $p(\theta)$, the operator $P/p(\mathbf{a})$ sends x^a to x^b, establishing (1).

Next let $P \in D(R)_{\mathbf{c}}$, and suppose that $\mathbf{a} \not\preceq \mathbf{a} + \mathbf{c}$. Then by Lemma 4.1.4 and the argument in the first paragraph, we have $P * x^{\mathbf{a}} = p(\mathbf{a}) x^{\mathbf{b}} = 0$. This proves (2). □

THEOREM 4.1.6. (1) *Let $\mathbf{a}_0 \in \mathbb{C}^d$. Then the set of simple subquotients of $x^{\mathbf{a}_0} \cdot R_A^{\pm}$ as a $D(R_A)$-module, is*

$$\{ \bigoplus_{\mathbf{b} \succeq \mathbf{a}} \mathbb{C}x^{\mathbf{b}} / \bigoplus_{\mathbf{b} \succ \mathbf{a}} \mathbb{C}x^{\mathbf{b}} \,|\, \mathbf{a} - \mathbf{a}_0 \in \mathbb{Z}A \}.$$

(2) *The set of simple subquotients of R_A as a $D(R_A)$-module, is*

$$\{ \bigoplus_{\mathbf{b} \succeq \mathbf{a}} \mathbb{C}x^{\mathbf{b}} / \bigoplus_{\mathbf{b} \succ \mathbf{a}} \mathbb{C}x^{\mathbf{b}} \,|\, \mathbf{a} \in \mathbb{N}A \}.$$

In particular, the number of simple subquotients equals the number of equivalence classes of parameters in $\mathbf{a}_0 + \mathbb{Z}A$ and in $\mathbb{N}A$, respectively.

PROOF. Proposition 4.1.5 implies that $\bigoplus_{\mathbf{b} \succeq \mathbf{a}} \mathbb{C}x^{\mathbf{b}}$ is the $D(R_A)$-submodule generated by $x^{\mathbf{a}}$, and that $\bigoplus_{\mathbf{b} \succ \mathbf{a}} \mathbb{C}x^{\mathbf{b}}$ is its maximal submodule. □

The correctness of the following algorithm follows from the correctness of Algorithm 3.4.2 and Theorem 4.1.6.

ALGORITHM 4.1.7. *Input: A.*
Output: the set of simple subquotients of R.

(1) *Take any set E_τ of complete representatives of the set $\mathbb{Q}(A \cap \tau) \cap \mathbb{Z}A / \mathbb{Z}(A \cap \tau)$.*
(2) *For each $\mathbf{l} \in E_\tau$, choose any $\mathbf{l}_+, \mathbf{l}_- \in \mathbb{N}^n$ with $\mathbf{l} = A\mathbf{l}_+ - A\mathbf{l}_-$.*
(3) *Choose any subset E'_τ of E_τ. Put $\mathcal{E}' := (E'_\tau)_\tau$.*
(4)
$$I(\mathcal{E}') = \bigcap_{\tau} \bigcap_{\mathbf{l} \in E'_\tau} \mathrm{mono}((\langle x^{\mathbf{l}_+} \rangle + I_A : x_j^\infty \, (\mathbf{a}_j \in \tau)) : x^{\mathbf{l}_-}).$$
(5)
$$C(\mathcal{E}') := I(\mathcal{E}') / \sum_{\mathcal{E}' \subsetneq \mathcal{E}''} I(\mathcal{E}'')$$
is a simple subquotient (though it could be trivial), where $\mathcal{E}' \subset \mathcal{E}''$ means ${E'}_\tau \subset {E''}_\tau$ for all τ. The set of simple subquotients of R is $\{ C(\mathcal{E}') \neq 0 \,|\, \mathcal{E}' \}$.

REMARK 4.1.8. For $I(\mathcal{E}')$ to be non-trivial, for $\tau \subset \tau'$, the image of E'_τ under the natural map from E_τ to $E_{\tau'}$ should be in $E'_{\tau'}$. Hence we should choose E'_τ from smaller faces.

EXAMPLE 4.1.9. Let

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix},$$

and consider the coordinate ring $R = R_A = \mathbb{C}[t_1t_2, t_1t_2^2, t_1^2, t_1^2t_2]$.

We compute $E_\tau(\mathbf{a})$ for all $\mathbf{a} \in \mathbb{N}A$ and all faces τ of $\mathbb{R}_{\geq 0}A$. Since $\mathbf{a}$ belongs to $\mathbb{N}A$, we see $E_\tau(\mathbf{a}) \ni 0$ for all faces τ, $E_{\{0\}}(\mathbf{a}) = \{0\}$, and $E_{\mathbb{R}_{\geq 0}A}(\mathbf{a}) = \{0\}$. Since $\mathbb{Q}(A \cap \sigma_1) \cap \mathbb{Z}A = \mathbb{Z}(A \cap \sigma_1)$, we see $E_{\sigma_1}(\mathbf{a}) = \{0\}$ as well. For the facet σ_2, we can take $\{0, {}^t(1,0)\}$ as a set of the complete representatives of $\mathbb{Q}(A \cap \sigma_2) \cap \mathbb{Z}A / \mathbb{Z}(A \cap \sigma_2)$. Hence $E_{\sigma_2}(\mathbf{a}) = \{0\}$ or $\{0, {}^t(1,0)\}$. Indeed

$$\begin{aligned} E_{\sigma_2}(\mathbf{a}) &= \{0, {}^t(1,0)\} && \text{if } a_2 \geq 1 \\ &= \{0\} && \text{if } a_2 = 0. \end{aligned}$$

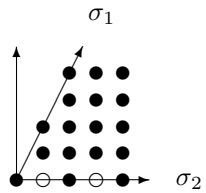


FIGURE 6. The semigroup $\mathbb{N}A$

The following is the composition series of R as a $D(R)$-module.

$$0 \subset \bigoplus_{\mathbf{a}\in\mathbb{N}A,\ a_2\geq 1} \mathbb{C}t^{\mathbf{a}} \subset \bigoplus_{\mathbf{a}\in\mathbb{N}A} \mathbb{C}t^{\mathbf{a}} = R.$$

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