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Invariant Theory and Differential Operators

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Constructive invariant theory was a preoccupation of many nineteenth century mathematicians, but the topic fell out of fashion in the early twentieth century. In the latter twentieth century the topic enjoyed a resurgence, partly due to its connections with the construction of moduli spaces in algebraic geometry and partly due to the development of computational algorithms suitable for implementation in modern symbolic computation packages. In this survey paper we briefly discuss some of the history and applications of invariant theory and apply one particular algorithm that uses Gröbner bases to find invariants of linearly reductive algebraic groups acting on the Weyl algebra. After showing how we can present the ring of invariant differential operators in terms of generators and relations, we turn to the operators on the invariant ring itself. The theory is particularly nice for finite groups acting on polynomial rings, but we also compute an example involving an $SL_2\mathbb{C}$-action. In this example, we give a complete description of the generators and relations of $D(G(2,4))$, the ring of differential

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operators on the Grassmannian of 2-planes in 4-space (or on the affine cone over the Grassmannian of lines in projective 3-space).

This paper is based on my talk in the workshop on Gröbner Bases in Symbolic Analysis held at RISC and RICAM in May 2006. Many of the technical details are omitted. The interested reader can find them in my paper [34] or in Derksen and Kemper's monograph [8], as indicated in the text.

1 Invariant Theory

When a group G acts on an affine algebraic variety X, then it makes sense to ask whether the orbits of G form an algebraic variety in their own right. This is the basic question at the heart of geometric invariant theory and the answer is subtle [24]. To make matters much easier, we restrict ourselves to the **non-modular case**: throughout this paper we work with complex varieties but all the results hold over any field of characteristic zero or in any situation where the characteristic of the field does not divide the order of a finite group G. Two simple examples suffice to introduce the theory.

Example 1.1 If $G = \mathbb{Z}_2 = \{-1, 1\}$ acts on the affine plane $X = \mathbb{A}^2_{\mathbb{C}}$ by scalar multiplication, $g \bullet (x, y) = (gx, gy)$, then all the orbits consist of two points except for the orbit of the origin, which is a fixed point of the group action. If the orbits do form an algebraic variety X/G then the natural projection map $X \to X/G$ that sends each point to its orbit is surjective and corresponds to an injective map of the coordinate rings $\mathbb{C}[X/G] \hookrightarrow \mathbb{C}[X] = \mathbb{C}[x, y]$. So $\mathbb{C}[X/G]$ can be identified with the subring of $\mathbb{C}[x, y]$ consisting of functions that are constant on each orbit. In our example, this just consists of those polynomials $f(x, y)$ such that $f(x, y) = f(-x, -y)$ and so $\mathbb{C}[X/G] = \mathbb{C}[x^2, xy, y^2] \cong \mathbb{C}[a, b, c]/(b^2 - ac)$. Though the space X and the G-action were about as nice as possible, the quotient variety X/G is a singular surface, a cone with vertex at the origin.

Generalizing this example, when G acts on a variety X there is a natural left action on $f \in R = \mathbb{C}[X]$ given by $(g \bullet f)(x) = f(g^{-1} \bullet x)$ and

$$R^G = \{f \in R : g \bullet f = f \text{ for all } g \in G\}$$

is the ring of G-invariant functions on X. The variety $X//G = \mathrm{Spec}(R^G)$ is called the categorical quotient of X by G. However, the categorical quotient may not be the quotient X/G as the next example demonstrates.

Example 1.2 If $G = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ acts on $X = \mathbb{A}^2_{\mathbb{C}}$ by scalar multiplication then most of the orbits have the form $L \setminus \{(0, 0)\}$, where L is a line in X passing through the origin. The sole exception is the orbit of the fixed point, $(0, 0)$. However, since any continuous function that is constant on an orbit must also take the same value on its closure, the fact that $(0,0)$ is in the closure of all orbits forces $\mathbb{C}[X//G] = \mathbb{C}$. That is, $X//G = \mathrm{Spec}(\mathbb{C})$ is a point. Since this doesn't seem to reflect the structure of the orbit space, the common approach is to restrict our attention to an open subset of points $Y \subset X$ on which G acts; these form a ringed space and on each affine chart $U \subset Y$ we can consider the ring of invariants $\mathbb{C}[U]^G$. Patching together the $\mathrm{Spec}(\mathbb{C}[U]^G)$ gives a variety Y/G. For instance, in the case of the torus acting on the plane, the algebraic variety $Y = X \setminus \{(0,0)\}$ is covered by two affine charts $Y_1 = \{(x,y) : \ x \neq 0\}$ and $Y_2 = \{(x,y) : \ y \neq 0\}$. Now it is not hard to see that on each chart the slope parameterizes the orbits – $Y_1/G = \mathrm{Spec}(\mathbb{C}[y/x])$ and $Y_2/G = \mathrm{Spec}(\mathbb{C}[x/y])$. Since these are also the charts for the projective line, $Y/G \cong \mathbb{P}^1_{\mathbb{C}}$.

Generalizing the method of the last example, we call a point x in a projective variety X semi-stable (and write $x \in X^{\mathrm{ss}}$) if there is an affine neighborhood U of x on which there is an invariant $f \in \mathbb{C}[U]^G$ such that $f(x) \neq 0$. The quotient $X^{\mathrm{ss}}//G$ is a projective variety, called the geometric invariant theory (G.I.T.) quotient of X under the G-action. It may still occur that the points in $X^{\mathrm{ss}}//G$ do not correspond to the orbits of G on X^{ss} (roughly speaking, the invariants may fail to separate orbits in X^{ss}), but even in this case, the variety $X^{\mathrm{ss}}//G$ enjoys many functorial properties that we would expect of a quotient. A trivial example of this construction occurs when G is finite; then every point in X is semi-stable and $X/G = X//G$. Here we ought to be clear that we are omitting many details of the G.I.T. construction. The interested reader is encouraged to consult [24] for the full story (or [9, chapters 6 and 8] for a cogent précis).

Let's look at some more complicated examples to further illustrate the power and applicability of the invariant theory viewpoint.

Example 1.3 One of the great tools in algebraic geometry is the construction of moduli spaces whose points parameterize varieties of interest. For instance, consider the variety $\mathcal{M}_{0,d}(\mathbb{P}^2)$ that parameterizes degree-d rational curves in the plane. Geometric invariant theory appears in the description of this space: we'd like to describe each curve using an explicit parametrization $\mathbb{P}^1 \to \mathbb{P}^2$ but then we need to identify those curves that differ only by a linear change of coordinates on the domain $\mathbb{P}^1$. To do this we take the quotient of the space of parameterizations by an $\mathrm{Aut}(\mathbb{P}^1) = PGL_2$-group action.

Example 1.4 Another important example of the G.I.T. method involves the construction of the Hilbert scheme parameterizing subvarieties of projective space with given Hilbert polynomial. A simple example is the Hilbert scheme parameterizing two points in $\mathbb{P}^1$, corresponding to the constant Hilbert polynomial with value 2. It is easy to parameterize pairs of points, just take $(a,b) \in \mathbb{P}^1 \times \mathbb{P}^1$. However, since the order of the points doesn't matter we should identify (a,b) with (b,a). Taking the quotient by the $\mathbb{Z}_2$-action that swaps the points, we obtain the Hilbert scheme for pairs of points in $\mathbb{P}^1$: $(\mathbb{P}^1 \times \mathbb{P}^1)/\mathbb{Z}_2$. Though it is a standard exercise in a first course in algebraic geometry to show that $\mathbb{P}^1 \times \mathbb{P}^1 \ncong \mathbb{P}^2$, it is less common to explain that once we quotient by the $\mathbb{Z}_2$ action we do get $\mathbb{P}^2$. Indeed, if we think of the points on $\mathbb{P}^1 \times \mathbb{P}^1$ as pairs of polynomials $(a_1x + a_2y, b_1x + b_2y)$ the multiplication map sends this pair to a degree two homogeneous polynomial, which is identified with an element of $\mathbb{P}^2$. The multiplication map is generically 2-to-1 but since we identify the pre-images of $(a_1x + a_2y)(b_1x + b_2y)$ in $(\mathbb{P}^1 \times \mathbb{P}^1)/\mathbb{Z}_2$ the induced map to $\mathbb{P}^2$ is an isomorphism. The book [25] contains a detailed exposition on Hilbert schemes.

Example 1.5 Another interesting example involves the Grassmannian $G(k, n)$, a variety whose points parameterize k-dimensional subspaces of n-space. Equivalently one could consider $\mathbb{G}(k-1, n-1) = \mathbb{P}(G(k, n))$, a projective variety parameterizing the projective $(k-1)$-dimensional linear spaces in $\mathbb{P}^{n-1}$. To describe each $(k-1)$-dimensional linear space $\mathbb{P}(V)$ in $\mathbb{P}^{n-1}$, we choose a basis $\{b_1, \ldots, b_k\}$ of V and associate to their span the k by n matrix whose rows consist of the b_i's. Among all k by n matrices, only the full-rank matrices correspond to $(k-1)$-dimensional spaces $\mathbb{P}(V)$, so we only consider the open set of $\mathbb{A}^{nk}$ consisting of full-rank matrices. Moreover, there are several parameterizations for each $\mathbb{P}(V)$, one for each choice of basis for $\mathbb{P}(V)$. To identify these copies we quotient by an SL_k-action, where SL_k acts on the k by n matrices by left multiplication. The quotient is precisely the Grassmannian $\mathbb{G}(k-1, n-1)$. The common way to describe this space is to compute $\mathbb{C}[X^n//SL_k\mathbb{C}] = \mathbb{C}[X^n]^{SL_k\mathbb{C}}$ where $X = \mathbb{C}^k$ and set $\mathbb{G}(k-1, n-1) = \mathbb{P}(X^n//SL_k\mathbb{C})$ (see [8, Section 4.4] or [32, Chapter 3] for details). In section 6 we compute the ring $\mathbb{C}[G(2, 4)] = \mathbb{C}[(\mathbb{C}^2)^4//SL_2\mathbb{C}]$ of functions on the Grassmannian $G(2, 4)$ and describe the ring of differential operators on $G(2, 4)$.

2 Structural Properties of Rings of Invariants

In general it is difficult to compute the ring of invariants $R^G = \mathbb{C}[X//G]$. Indeed, this was a major field of research for mathematicians in the nineteenth century. In 1868 the acknowledged "king of invariant theory" Paul Gordan proved that when $G = SL_2\mathbb{C}$ acts on a finite dimensional $\mathbb{C}$-vector space X, the ring of invariants R^G is a finitely generated $\mathbb{C}$-algebra. Moreover, his proof was constructive so that – at least in principle – it was possible to compute a set of generators. In 1890 David Hilbert stunned the mathematical community by giving a nonconstructive proof that whenever a linearly reductive group G acts on a finite dimensional $\mathbb{C}$-vector space, the ring of invariants R^G is a finitely generated $\mathbb{C}$-algebra. Hilbert's nonconstructive proof met with serious opposition. Gordan even described it as "Theologie und nicht Mathematik!". Hilbert continued to consider invariant theory a major area of mathematics: his 14th problem [12] is related to the question of whether R^G is finitely generated for any group acting on a finite dimensional vector space. Masayoshi Nagata answered this question – and Hilbert's 14th problem – in the negative [26], providing an example where G is not linearly reductive and R^G fails to be finitely generated. For details, see the expository article [23].

In today's mathematical culture it may seem hard to believe that nonconstructive methods like those used by Hilbert met with such fierce resistance. Perhaps in order to counter his critics, Hilbert provided a constructive method to compute the generators for R^G just three years after the publication of his controversial proof [11]. However, nearly a hundred years went by before Harm Derksen turned Hilbert's ideas into something that could actually be used for symbolic computation. We'll describe Derksen's algorithm in the next section. For now, let's examine Hilbert's proof that R^G is finitely generated. The proof depends on a certain map $\mathcal{R} : R \to R^G$ called the Reynolds operator.

Recall that an algebraic group G is called linearly reductive if every G-invariant subspace W of a G-vector space V has a G-invariant complement: $V = W \oplus W^C$. Examples of linearly reductive groups in characteristic zero are GL_n, all semi-simple groups including SL_n, O_n and Sp_n, finite groups and tori. Finite groups are also linearly reductive in prime characteristic when the characteristic does not divide the order of the group. Now let a linearly reductive group G act on a finite dimensional vector space X. Since the induced G-action on $R = k[X]$ preserves degree we see that the inclusion $R^G \hookrightarrow R$ is a graded map of R^G-algebras. Restricting to the degree d piece, the G-invariant subspace R_d^G of R_d has a G-invariant complement and for each d we can project R_d onto R_d^G. The Reynolds operator $\mathcal{R}$ is the R^G-linear map $R \to R^G$ that agrees with this projection in each degree. Note that the Reynolds operator is a splitting of the inclusion $R^G \to R$ as a map of R^G-algebras.

In general it can be quite difficult to compute the Reynolds operator for a given group action. However, when G is a finite group the Reynolds operator just averages the group action:

$$\mathcal{R}(f) = \frac{1}{|G|} \sum_{g \in G} g \bullet f.$$

When G is infinite then we can compute the Reynolds operator by integrating over a compact subgroup. In particular when G is a connected semi-simple group there are explicit algebraic algorithms [8, Algorithm 4.5.19] to compute the value of the Reynolds operator on any element of R, though no simple closed form algebraic expression for $\mathcal{R}$ is known in these cases. In the special case of $G = SL_n$ or $G = GL_n$, Cayley's Omega process does give a closed form expression for the Reynolds operator (see [8, section 4.5.3]).

Theorem 2.1 (Hilbert (1890)) *If G is a linearly reductive group acting on a Noetherian k-algebra R, then R^G is a finitely generated k-algebra.*

Proof. Let I be the Hilbert ideal of R, the ideal generated by all the G-invariant functions of positive degree: $I = (f \in R^G_{>0})R$. Since R is Noetherian, I is a finitely generated ideal in R. Moreover, I is a homogeneous ideal, so we can find homogeneous elements $f_1, \ldots, f_t$ in $R^G_{>0}$ generating the R-ideal I. Now $k[f_1, \ldots, f_t] \subseteq R^G$, but we claim that we actually have equality. We prove this for each graded piece of R^G by induction. The base case is trivial since $k[f_1, \ldots, f_t]_0 = R^G_0 = k$. Now assume that the rings agree in degree less than d and let $g \in R^G_d$. Then $g \in I$ so there exist homogeneous elements $h_1, \ldots, h_t$ of R such that $\deg(h_i) = d - \deg(f_i) < d$ and

$$g = h_1 f_1 + \cdots + h_t f_t.$$

Applying the R^G-linear Reynolds operator $\mathcal{R}$ gives

$$g = \mathcal{R}(g) = \mathcal{R}(h_1)f_1 + \cdots + \mathcal{R}(h_t)f_t. \qquad (2.1)$$

Now since $\mathcal{R}(h_i) \in R^G$ has degree less than d, $\mathcal{R}(h_i) \in k[f_1, \ldots, f_t]$. Now (2.1) shows that $g \in k[f_1, \ldots, f_t]$. This completes the inductive step so $R^G = k[f_1, \ldots, f_t]$. □

It is possible to use the Reynolds operator, together with the theory of tight closure, to give an elegant proof [14, Theorem 3.6] of a theorem due to Hochster and Roberts [13].

Theorem 2.2 *If a linearly reductive group G acts on a Noetherian $\mathbb{C}$-algebra R, then R^G is Cohen-Macaulay. That is, there is a homogeneous system of parameters $f_1, \ldots, f_d$ in R^G such that $\mathbb{C}[f_1, \ldots, f_d]$ is a polynomial ring, and R^G is a finite $\mathbb{C}[f_1, \ldots, f_d]$-module. The parameters f_i are said to be primary invariants and the module generators are called secondary invariants.*

Finding primary and secondary invariants tends to require significant computation, but the amount of computation is reduced if we know the number and degree in which these invariants occur. This is precisely the information contained in the classical statement of Molien's theorem, which deals with finite group actions.

If G is a group acting on $R = \mathbb{C}[x_1, \ldots, x_n]$, then the Molien series is the Hilbert series for the ring R^G, a series that encodes the dimensions of the graded pieces of R^G:

$$H(R^G, t) = \sum_{d=0}^{\infty} \left(\dim_{\mathbb{C}} R^G_d\right) t^d.$$

In 1897 Molien proved that it is possible to compute $H(R^G, t)$ without first computing R^G.

Theorem 2.3 (Molien's Theorem) *If G is a finite group of order $|G|$ acting on $R = \mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_n]$ via the representation $\rho : G \to GL(V)$ then the Molien series can be expressed as*

$$H(R^G, t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{det(1 - t\rho(g))}.$$

We refer the reader to Sturmfels's account [32, Theorem 2.2.1] for a very readable proof that only relies on elementary linear algebra. Replacing the sum by an integral, Molien's theorem can be extended to algebraic groups (see [8] for details).

The Molien series can be expressed in the form

$$H(R^G, t) = \frac{P(t)}{\prod_{i=1}^{p}(1 - t^{d_i})}.$$

The degrees d_i of the primary invariants can be read off this expression, as can the degrees k_i and number in each degree m_i of the secondary invariants: these are encoded by the polynomial $P(t) = \sum m_i t^{k_i}$. There are algorithms to compute the primary invariants (see [5]). Once these are found, we can apply the Reynolds operator to a basis for R_d until the results (together with the polynomials of degree d in the polynomial algebra generated by the primary invariants) span a vector space of dimension $\dim_{\mathbb{C}}(R_d^G)$, as predicted by the Molien series.

We end this section with a short example to illustrate Molien's theorem.

Example 2.4 Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} = \langle \gamma, \delta : \gamma^2 = \delta^4 = \mathrm{id}_G \rangle$ act on $X = \mathbb{C}^3$ so that γ is a reflection in the x_2x_3-plane and δ is a 90-degree rotation about the x_1-axis: the representation $\rho : G \to End_{\mathbb{C}}(\mathbb{C}^3)$ is given by $\rho(\gamma) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\rho(\delta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$. The Molien series is

$$H(R^G, t) = \frac{1 + t^4}{(1 - t^2)^2(1 - t^4)}.$$

It is not hard to see that $x_1^2, x_2^2 + x_3^2, x_2^4 + x_3^4$ is a system of parameters of the degrees required by the Molien series. These form the primary invariants. There is a single secondary invariant in degree 4. Using the Reynolds operator, we find the secondary invariant to be $x_2x_3^3 - x_2^3x_3$. We will return to this example throughout the paper.

3 Computing Rings of Invariants

There are a variety of algorithms to compute rings of invariants. One of the oldest is Gordan's symbolic calculus [27], which deals with the important case where $G = SL_n(\mathbb{C})$ acts on n-ary d-forms. Cayley's Omega process [32] uses differential operators to compute invariants ([8, section 4.5.3],[32, section 4.3]) and when G is a Lie group, we also have access to infinitesimal methods[1] based on the induced Lie algebra action [32, section 4.3]. Additionally, in many circumstances we can use Molien's theorem to help search for generators, as described above. If we can find a homogeneous system of parameters for R^G to serve as the primary invariants then we can reduce the problem of finding the secondary invariants to a large linear algebra problem. This is a very appealing approach but it is not always easy to find a set of primary invariants. Kemper [16] gives a good exposition describing many methods to compute rings of invariants (also see [6]).

[1] Recently Bedratyuk [1, 2] produced invariants and co-variants for binary forms in previously inaccessible cases by solving the differential equations coming from the infinitesimal action of $SL_2(\mathbb{C})$. These very interesting papers are only peripherally related to the material in this paper but they are highly recommended.

Instead of describing these approaches, we return to Hilbert's original construction of the finite set of generators. This algorithm was generally dismissed as being far too computationally expensive, but in 1999 Harm Derksen surprised many mathematicians by finding an elegant way to recast Hilbert's ideas into a simple algorithm [7]. Though other algorithms may be faster than Derksen's algorithm, it is appealing because it can be applied in a wide variety of contexts. We choose to describe it in detail since it uses Gröbner bases and fits in well with the theme of these conference proceedings.

Let G be a linearly reductive group acting on a vector space $X = \operatorname{Spec}(R)$. Derksen's algorithm is based on the observation that the zero set of the Hilbert ideal I of $R = \mathbb{C}[x_1, \ldots, x_n]$, the ideal generated by all positive degree invariants, is precisely the non-semi-stable points of X (see [8, Lemma 2.4.2]). The collection of these points $\mathbb{V}(I) = X \setminus X^{\mathrm{ss}}$ is called the nullcone of X and denoted $\mathcal{N}_X$. To describe the algorithm we first parameterize G so that we can think of G as an algebraic variety. If G is a finite group then we can identify the elements of G with a finite set of points and if G is an algebraic group then this parametrization is implicit in the definition of G. Now let $\psi : G \times X \to X \times X$ be the map of varieties given by $\psi(g, x) = (x, g \bullet x)$. Let Z be the image of ψ and let $\overline{Z}$ be its Zariski-closure. Identify $\mathbb{C}[X \times X]$ with $\mathbb{C}[\mathbf{x}, \mathbf{y}] = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Now we claim that

$$\overline{Z} \cap (X \times \{0\}) = \mathcal{N}_X \cap \{0\}. \tag{3.1}$$

If $(w, 0) \in \mathcal{N}_X \cap \{0\}$ then w is not a semi-stable point and so $0 \in \overline{Gw}$. Thus $(w, 0)$ is in the closure of the image Z. For the other inclusion, we prove that if $(w, 0) \in \overline{Z}$ then $w \in \mathbb{V}(I) = \mathcal{N}_X$. Suppose that $f \in I$ has positive degree. Then $f(\mathbf{x}) - f(\mathbf{y})$ vanishes on all of Z because $f(x) - f(g \cdot x) = 0$. But then $f(\mathbf{x}) - f(\mathbf{y})$ must also vanish on the closure of Z. In particular, $f(w) - f(0) = f(w) = 0$. Thus $w \in \mathbb{V}(I)$, as desired.

Derksen [8, Theorem 4.1.3] used the Reynolds operator to show that the equality (3.1) of sets actually descends to an equality of ideals. If $B = \mathbb{I}(\overline{Z})$ then

$$B + (y_1, \ldots, y_n) = I + (y_1, \ldots, y_n).$$

Now we can compute the ideal B by elimination using Gröbner basis methods and then setting each of $y_1, \ldots, y_n$ to zero we get the generators for the ideal I.

These observations lead to the following algorithm to compute R^G:

Algorithm 3.1 (Derksen's algorithm) INPUT: A linearly reductive algebraic group G acting on a finite dimensional complex vector space X by the representation ρ.
OUTPUT: A generating set for $\mathbb{C}[X]^G$.
STEP 1: Parameterize the group G by the zero set of an ideal $J \subset \mathbb{C}[\mathbf{t}] = \mathbb{C}[t_1, \ldots, t_k]$. As well, express the representation ρ in as a matrix A whose entries are polynomials in $\mathbb{C}[\mathbf{t}]$.
STEP 2: Construct the ideal $\mathbb{I}(\Gamma)$ describing the graph Γ of $\psi : G \times X \to X \times X$ as follows. Identify the first copy of X in the range with the copy of X in the domain and, writing $\mathbf{x}$ for the column vector containing the variables $x_1, \ldots, x_n$, construct the ideal

$$\mathbb{I}(\Gamma) = (y_1 - (A\mathbf{x})_1, \ldots, y_n - (A\mathbf{x})_n) + J\mathbb{C}[\mathbf{t}, \mathbf{x}, \mathbf{y}]$$

in the ring $\mathbb{C}[\mathbf{t},\mathbf{x},\mathbf{y}]$.
STEP 3: Compute a Gröbner basis for $I(\Gamma)$ in an elimination order on $\mathbb{C}[\mathbf{t},\mathbf{x},\mathbf{y}]$ that gives the parameters $\mathbf{t}$ higher weight that the $\mathbf{x}$'s and $\mathbf{y}$'s (see [4] or [17] for details on elimination). Intersecting this basis with $\mathbb{C}[\mathbf{x},\mathbf{y}]$ gives generators for the ideal B.
STEP 4: Set $y_1 = \cdots = y_n = 0$ to get generators for the Hilbert ideal I of R.
STEP 5: The generators from step 4 may fail themselves to be invariants. So apply the Reynolds operator to each of them to get invariants that generate the Hilbert ideal I. These invariants also generate the ring R^G, as described in Theorem 2.

Example 3.2 Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} = \langle < \gamma, \delta : \gamma^2 = \delta^4 = \mathrm{id}_G \rangle$ act on $X = \mathbb{C}^3$ as in Example 2.4 so that γ is a reflection in the x_2x_3-plane and δ is a 90-degree rotation about the x_1-axis. We parameterize G by the pairs (s,t) where s is a square root of 1 ($s = -1$ corresponds to γ) and t is a fourth root of 1 ($t = i$ corresponds to δ). Then interpolating the representation matrices gives a parametrization of the representation,

$$\rho(s,t) = \begin{bmatrix} s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{t^3+t}{2} & \frac{(t-t^3)i}{2} \\ 0 & \frac{(t^3-t)i}{2} & \frac{t^3+t}{2} \end{bmatrix} = \begin{bmatrix} s & 0 & 0 \\ 0 & \frac{t^3+t}{2} & \frac{(t-t^3)i}{2} \\ 0 & \frac{(t^3-t)i}{2} & \frac{t^3+t}{2} \end{bmatrix}.$$

We compute the ring of invariants using Derksen's algorithm. We write $\mathbb{I}(\Gamma) = (s^2 - 1, t^4 - 1, y_1 - (sx_1), y_2 - (\frac{t^3+t}{2}x_2 + \frac{(t^3-t)i}{2}x_3), y_3 - (\frac{(t-t^3)i}{2}x_2 + \frac{t^3+t}{2}x_3))$ and compute a Gröbner basis in an elimination order designed to eliminate s and t. For example, we can use a product order, refined by degree lex order $\prec$, in which the first block of variables is $s \prec t$ and the second block of variables is $x_1 \prec x_2 \prec x_3 \prec y_1 \prec y_2 \prec y_3$. The Gröbner basis $\mathcal{G}$ contains 22 polynomials. Considering only $\mathcal{G} \cap \mathbb{C}[x_1,x_2,x_3,y_1,y_2,y_3]$ gives seven polynomials and setting $y_1 = y_2 = y_3 = 0$ kills 3 of these, leaving $\{x_2^2+x_3^2, x_1^2, x_3^4, x_2x_3^3+ix_3^4\}$. Applying the Reynolds operator to these four polynomials produces a Gröbner basis for the Hilbert ideal I: $I = (x_2^2+x_3^2, x_1^2, x_2^4+x_3^4, ix_2^4 - x_2^3x_3 + x_2x_3^3 + ix_3^4)$. Cleaning this up shows that $I = (x_2^2+x_3^2, x_1^2, x_2^4+x_3^4, x_2^3x_3 - x_2x_3^3)$. So $\mathbb{C}[x_1,x_2,x_3]^G = \mathbb{C}[x_2^2+x_3^2, x_1^2, x_2^4+x_3^4, x_2^3x_3 - x_2x_3^3]$, as in Example 2.4. Now another elimination computation shows that the quotient variety is a singular hypersurface: setting $a = x_2^2+x_3^2$, $b = x_1^2$, $c = x_2^4+x_3^4$ and $d = x_2^3x_3 - x_2x_3^3$ gives

$$\mathbb{C}[x_1,x_2,x_3]^G \cong \mathbb{C}[a,b,c,d]/(a^4 - 3b^2c + 2c^2 + 2d^2).$$

Note that the singularities lie along the line $a = c = d = 0$, which corresponds to the quotient of the x_1-axis by the group action.

4 Group Actions on the Weyl Algebra

The Weyl algebra is the algebra of differential operators on affine n-space. It can be used to formulate quantum mechanics (see [3]) and to study systems of differential equations in an algebraic manner (see, for example, [30]). To be precise, if

$R = \mathbb{C}[x_1, \ldots, x_n]$ is the coordinate ring of $X = \mathbb{A}^n_{\mathbb{C}}$ then the Weyl algebra $D(R)$ is the ring $\mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$ in which the variables $x_1, \ldots, x_n$ commute among themselves, the variables $\partial_1, \ldots, \partial_n$ commutate among themselves, and the ∂_i's and the x_j's interact via the commutator relation $[\partial_i, x_j] := \partial_i x_j - x_j \partial_i = \delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. The variables ∂_i should be thought of as the operators $\partial/\partial x_i$ on the ring $R = \mathbb{C}[x_1, \ldots, x_n]$ and the variables $x_j \in D(R)$ should be thought of as the operators that multiply functions in R by x_j. Under this interpretation the rule for commuting ∂_i and x_i corresponds to the product rule in multi-variable calculus:

$$\begin{aligned}
(\partial_i x_i) \bullet f(x_1, \ldots, x_n) &= \tfrac{\partial}{\partial x_i}(x_i f(x_1, \ldots, x_n)) \\
&= x_i \tfrac{\partial f}{\partial x_i}(x_1, \ldots, x_n) + f(x_1, \ldots, x_n) \\
&= (x_i \partial_i + \delta_{ii}) \bullet f(x_1, \ldots, x_n).
\end{aligned}$$

When G acts on affine space $X = \mathbb{A}^n_{\mathbb{C}}$, it not only induces an action on the coordinate ring $R = \mathbb{C}[X]$ but also on the Weyl algebra $D(R)$. For $g \in G$, $\theta \in D(R)$ and $f \in R$,

$$(g \bullet \theta)(f) = g \bullet (\theta(g^{-1} \bullet f)).$$

Those readers familiar with differential geometry will not find it surprising that G acts on the operators $\partial_1, \ldots, \partial_n$ via the contragredient representation: if $g \in G$ acts on $R = \mathbb{C}[x_1, \ldots, x_n]$ via the matrix A, $g \bullet \mathbf{x} = A\mathbf{x}$ (all vectors are represented by column matrices), then g acts on $\mathbb{C}[\partial] = \mathbb{C}[\partial_1, \ldots, \partial_n]$ via $\left(A^T\right)^{-1}$, where T stands for the Hermitian transpose. However, many readers might enjoy an explicit proof of this fact communicated to the author by Harrison Tsai. To establish this claim, it is enough to show that the defining identities $[\partial_i, x_j] := \partial_i x_j - x_j \partial_i = \delta_{ij}$ for the Weyl algebra $D(R)$ are preserved under the proposed group action.

We first observe that the identities can be written in the matrix formulation

$$[\partial, \mathbf{x}] = \partial \mathbf{x}^T - (\mathbf{x}\partial^T)^T = \mathbf{1}.$$

At first sight this may seem odd because we are familiar with the formula $(AB)^T = B^T A^T$ in $GL_n(\mathbb{C})$ but such a formula depends on the commutativity of multiplication in $\mathbb{C}$, while here the x's and the ∂'s do not commute.

Now we show that the identity is preserved under the group action. For ease of notation, let B stand for $\left(A^T\right)^{-1}$, then

$$\begin{aligned}
[g \bullet \partial, g \bullet \mathbf{x}] &= (g \bullet \partial)(g \bullet \mathbf{x}) - (g \bullet \mathbf{x})(g \bullet \partial) \\
&= B\partial\,(A\mathbf{x})^T - \left(A\mathbf{x}\,(B\partial)^T\right)^T \\
&= B\partial\mathbf{x}^T A^T - \left(A\mathbf{x}\partial^T B^T\right)^T \\
&= B\partial\mathbf{x}^T A^T + B\left(-\mathbf{x}\partial^T\right)^T A^T \\
&= B\partial\mathbf{x}^T A^T + B\left(\mathbf{1} - \partial\mathbf{x}^T\right) A^T \\
&= B\left(\partial\mathbf{x}^T + \mathbf{1} - \partial\mathbf{x}^T\right) A^T \\
&= B\mathbf{1}A^T \\
&= (A^T)^{-1} A^T = \mathbf{1}.
\end{aligned}$$

In an earlier paper [34] it was shown how to extend Derksen's algorithm to the Weyl algebra in order to compute the ring of invariant differential operators $D(R)^G$. For this we exploit the close connection between $D(R)$ and the commutative ring $GrD(R)$. To introduce $GrD(R)$, note that $D(R)$ is a filtered ring: if we assign degree 1 to each ∂_i and degree 0 to each x_j we say that an operator in $D(R)$ has order $\leq n$ if some representation of the operator has degree no greater than n. Note that it we need to be cautious when determining the order of an operator: for example, the operator $1 = \partial_i x_i - x_i \partial_i$ seems to have order 0 or 1, depending on its representation. Of course, 1 is an operator of order ≤ 0. If F_n consists of those operators of order $\leq n$, it is immediate that (1) $F_n \subset F_{n+1}$, (2) F_n is closed under addition and (3) $F_n \cdot F_m \subset F_{n+m}$. These properties ensure that the F_n define a filtration on the algebra $D(R) = \cup_{n\geq 0} F_n$.

Whenever we have a filtered ring such as $D(R)$, we can form its graded ring,

$$GrD(R) = \bigoplus_{n\geq 0} \frac{F_n}{F_{n+1}}.$$

The graded ring comes equipped with a symbol map, $\sigma : D(R) \to GrD(R)$, assigning $\sigma(\theta) = \theta \bmod F_{n+1}$ to each $\theta \in F_n$. If we write ξ_i for $\sigma(\partial_i)$ (and abuse notation by writing x_j for $\sigma(x_j) \in GrD(R)$ too) it is easy to see that $GrD(R)$ is generated by $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$. Moreover, $GrD(R)$ is a commutative ring since the commutation relation $\partial_i x_j - x_j \partial_i = \delta_{ij}$ in $D(R)$ becomes $\xi_i x_j - x_j \xi_i = 0 \bmod F_0$ in $GrD(R)$. Indeed, this shows that $GrD(R) = \mathbb{C}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$ is a polynomial ring in $2n$ variables.

The group G preserves the order filtration when it acts on $D(R)$, so there is an induced action on the graded ring $GrD(R)$. Indeed, the action of G on $D(R)$ is compatible with the symbol map, so if $g \in G$ acts on $x_1, \ldots, x_n$ via the matrix A then as in $D(R)$, g acts on $\xi_1, \ldots, \xi_n$ via the matrix $(A^T)^{-1}$. Moreover, the filtration $\{F_n\}$ restricts to a filtration on R^G, giving rise to the graded ring $Gr(D(R)^G)$. Since the action is compatible with the filtration, it should come as no surprise that $Gr(D(R)^G) = [GrD(R)]^G$; see [34, Theorem 1] for a proof.

Now we can apply Derksen's algorithm to the polynomial ring $GrD(R)$ to compute $[GrD(R)]^G = Gr(D(R)^G)$. Then we can lift the generators of $Gr(D(R)^G)$ to elements of $D(R)^G$. It is not hard to prove that if S is a filtered $\mathbb{C}$-algebra then any lifting of a set of generators for GrS is a set of generators for S, so the lifts of the generators of $Gr(D(R)^G)$ generate the ring of invariant differential operators $D(R)^G$.[1]

Example 4.1 We compute generators for $D(R)^G$ where G and R are as in Example 3.2. Listing the generators for $GrD(R)$ in the order $x_1, x_2, x_3, \xi_1, \xi_2, \xi_3$, the action of

[1] It is possible to simplify the previous discussion using the Poincare-Birkhoff-Witt theorem on normal orderings in $D(R)$; however, it is not clear how to apply Derksen's algorithm directly to $D(R)$, so we've taken a more elementary approach in this paper.

G on $GrD(R)_1$ is given by $\tilde{\rho} : G \to Aut_{\mathbb{C}}(GrD(R)_1)$, where

$$\tilde{\rho}(s,t) = \rho(s,t) \oplus \rho(s,t) = \begin{bmatrix} s & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{t^3+t}{2} & \frac{(t-t^3)i}{2} & 0 & 0 & 0 \\ 0 & \frac{(t^3-t)i}{2} & \frac{t^3+t}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & s & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{t^3+t}{2} & \frac{(t-t^3)i}{2} \\ 0 & 0 & 0 & 0 & \frac{(t^3-t)i}{2} & \frac{t^3+t}{2} \end{bmatrix}.$$

Following Derksen's algorithm, we write $\mathbb{I}(\Gamma) = (s^2-1, t^4-1, y_1 - (sx_1),$ $y_2 - (\frac{t^3+t}{2}x_2 + \frac{(t^3-t)i}{2}x_3), y_3 - (\frac{(t-t^3)i}{2}x_2 + \frac{t^3+t}{2}x_3), \eta_1 - (s\xi_1), \eta_2 - (\frac{t^3+t}{2}\xi_2 + \frac{(t^3-t)i}{2}\xi_3),$ $\eta_3 - (\frac{(t-t^3)i}{2}\xi_2 + \frac{t^3+t}{2}\xi_3))$ and compute a Gröbner basis in an elimination order designed to eliminate s and t. The Gröbner basis $\mathcal{G}$ consists of 92 polynomials. But $\mathcal{G} \cap \mathbb{C}[x_1, x_2, x_3, y_1, y_2, y_3, \xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3]$ consists of only 48 polynomials. After setting $y_1 = y_2 = y_3 = \eta_1 = \eta_2 = \eta_3 = 0$, we recover only 17 polynomials and applying the Reynolds operator to these gives seventeen generators for $Gr(D(R)^G)$. Replacing ξ_i with ∂_i and clearing fractions, et cetera, we get the following seventeen generators for the ring of invariant differential operators $D(R)^G$:

$$\left\{ \begin{array}{lll} \partial_2^2 + \partial_3^2, & x_3\partial_2 - x_2\partial_3, & x_2\partial_2 + x_3\partial_3, \\ \partial_1^2, & x_1\partial_1, & -x_2x_3\partial_2^2 + x_2x_3\partial_3^2, \\ x_1^2, & \partial_2^4 + \partial_3^4, & -\partial_2^3\partial_3 + \partial_2\partial_3^3, \\ x_2\partial_2^3 + x_3\partial_3^3, & -x_3\partial_2^3 + x_2\partial_3^3, & x_2^2\partial_2^2 + x_3^2\partial_3^2, \\ x_2^2 + x_3^2, & x_2^3\partial_2 + x_3^3\partial_3, & -x_2^2x_3\partial_2 + x_2x_3^2\partial_3, \\ x_2^4 + x_3^4, & -x_2^3x_3 + x_2x_3^3 & \end{array} \right\}.$$

It is worth noting that the Molien series for $(GrD(R))^G$ is

$$\frac{1 + 2t^2 + 10t^4 + 2t^6 + t^8}{(1-t^2)^5(1-t^4)}.$$

Thus, it requires 16 secondary generators to generate $(GrD(R))^G$ as a module over a polynomial ring generated by 6 primary invariants. In this example, Derksen's algorithm finds fewer generators of $(GrD(R))^G$ and $D(R)^G$ than Molien's method, but they are *algebra generators* rather than *module generators*. Perhaps this trade-off is inevitable: we seem to need a larger number of generators if we require them to enjoy better structural properties.

Not only can we compute the generators for $D(R)^G$, but we can also compute the relations among these generators. Using elimination we can compute the relations among the generators of $[GrD(R)]^G = Gr(D(R)^G)$. Each of these can be lifted to a relation in $D(R)^G$ (see below for an example). The complete set of relations among the generators in $D(R)^G$ is the two-sided ideal of $D(R)^G$ generated by these lifted relations and the commutator relations among the generators. For details, see [34, Algorithm 10]).

Example 4.2 We continue Example 4.1 and find the relations among the 17 generators of $D(R)^G$. To start we perform an elimination computation to compute the relations among the 17 generators of $Gr(D(R)^G)$. This computation is surprisingly fast (under 6 seconds on a Pentium III 933 MHz computer with 376 MB of RAM), but yields 221 relations among the generators in the graded ring. For instance, one of these relations indicates that

$$(x_1\xi_1)^2 - (x_1^2)(\xi_1^2) = 0 \text{ in } Gr(D(R)^G).$$

This only means that $(x_1\partial_1)^2 - (x_1^2)(\partial_1^2)$ is an operator of order less than 2 in $D(R)^G$. Performing the computation in the Weyl algebra[2] the operator equals $x_1\partial_1$ so the graded relation lifts to the relation

$$(x_1\partial_1)^2 - (x_1^2)(\partial_1^2) - (x_1\partial_1) = 0 \text{ in } D(R)^G.$$

The 221 lifted relations, together with the commutator relations among the 17 generators (there are 82 nontrivial commutator relations) generate the two-sided ideal in $D(R)^G$ of relations among the given generators.

At this stage the reader might well wonder whether there is a smaller set of generators for the ring $D(R)^G$. In fact, Levasseur and Stafford [19] prove that for finite groups G, the ring $D(R)^G$ is generated *as a noncommutative algebra* by the operators generating $\mathbb{C}[x_1,\ldots,x_n]^G$ and the operators generating $\mathbb{C}[\partial_1,\ldots,\partial_n]^G$. This greatly reduces the number of generators, but at the moment there is no good way to determine the relations among these generators. As well, the symbols of the generators that Levasseur and Stafford provide are not sufficient to generate $Gr(D(R)^G)$. There seems to be a need for a noncommutative version of Derksen's algorithm – one that works directly in $D(R)$ and not through $GrD(R)$ – though it remains an open problem to generalize Derksen's work in this direction.

5 Rings of Differential Operators

Alexander Grothendieck [10] introduced rings of differential operators associated to algebraic varieties. Suppose that $X \subset \mathbb{A}^n_{\mathbb{C}}$ is an algebraic variety and that X is the vanishing set of the ideal $I \subset R = \mathbb{C}[x_1,\ldots,x_n]$. Then the ring of differential operators can be described in terms of the Weyl algebra $D(R)$ (see [21, Chapter 15] for details):

$$D(X) := D(R/I) := \frac{\{\theta \in D(R) : \theta \bullet I \subseteq I\}}{ID(R)}.$$

The ring $D(X)$ inherits a filtration from the ring $D(R)$ and, just as for the Weyl algebra, $GrD(X)$ is a commutative ring (see [21] or [22] for a nice explanation of these facts).

[2] Many computer algebra systems can compute in the Weyl algebra: in SINGULAR we can use the PLURAL package; in Macaulay2 we can use the Dmodules package; in MAPLE we can use the Ore algebra package and both RISA/ASIR and CoCoA also support such computations.

The rings $D(R)$ have been the subject of intense study for many years. Levasseur and Stafford's monograph [18] is a good description of rings of differential operators and their connection to invariant theory.

We apply these definitions to the case where $X = \mathbb{A}^n_{\mathbb{C}}//G$ for some linearly reductive group G. We first realize X as an embedded variety in an affine space $\mathbb{A}^d_{\mathbb{C}}$ by presenting the ring $R^G = \mathbb{C}[X]$ as a finitely generated algebra: $R^G \cong \mathbb{C}[t_1, \ldots, t_n]/J$. Then $D(R^G) = \{\theta \in D(\mathbb{C}[t_1, \ldots, t_d]) : \theta \bullet J \subseteq J\}/JD(\mathbb{C}[t_1, \ldots, t_d])$. Now we need to be cautious: the ring of differential operators on the quotient variety is not the same thing as the ring of invariant differential operators! However, the natural map $\pi : X \to X//G$ induces the inclusion $R^G \hookrightarrow R = \mathbb{C}[\mathbb{A}^n_{\mathbb{C}}] = \mathbb{C}[x_1, \ldots, x_n]$ and in turn this induces a map $\pi_* : D(R)^G \to D(R^G)$ given by restriction. To be precise, if $\theta \in D(R)^G$, then $\pi_*(\theta)$ is the map that makes the diagram commute.

$$\begin{tikzcd} R \arrow[r, "\theta"] & R \arrow[d, two heads, "\mathcal{R}"] \\ R^G \arrow[u, hook, "i"] \arrow[r, dotted, "\pi_*\theta"'] & R^G \end{tikzcd}$$

If $\theta \in D(R)^G$, and $r \in R^G$ then $(\pi_*\theta)(r) = \theta(r)$. We check that $\theta(r) \in R^G$: for any $g \in G$,

$$g \bullet (\theta(r)) = g \bullet (\theta(g^{-1} \bullet r)) = (g \bullet \theta)(r) = \theta(r).$$

In general $\pi_*\theta = \mathcal{R}(\theta \circ i)$ is a differential operator on R^G of no higher order than θ.

5.1 Finite group actions

We turn to the case of a finite group action on a polynomial algebra over a characteristic zero field.

Theorem 5.1 (Kantor [15], Levasseur [20]) *When G is a finite group acting on a polynomial ring R, the map $\pi_* : D(R)^G \to D(R^G)$ is injective.*

Proof. See [34, Theorem 2]. □

Example 5.2 The map π_* can fail to be surjective. Consider the group $G = \mathbb{Z}_2$ acting on $R = \mathbb{C}[x]$ so that the generator of G sends x to $-x$. Then $R^G = \mathbb{C}[x^2]$ is a polynomial ring and $D(R^G)$ is a Weyl algebra. However, $D(R)^G = \mathbb{C}\langle x^2, x\partial, \partial^2\rangle$ so $D(R)^G$ is not isomorphic to $D(R^G)$. Thus π_* is not a surjection. Schwarz [31, Example 5.7] gives a more detailed argument.

The group action in Example 5.2 was generated by a reflection. We say that an element $g \in G$ acts as a pseudoreflection if it acts on X such that the eigenvalues of the action of g are all 1 except for a single value (which must be a root of unity since G is assumed to be finite). Equivalently, $g \in G$ is a pseudoreflection when the action of G fixes (pointwise) a codimension 1 hypersurface; in our case, the fixed set is a hyperplane since the action of G is linear. We call a group a reflection group if G is generated by pseudoreflections. The celebrated Sheppard-Todd-Chevalley theorem shows that $D(R^G)$ is a Weyl algebra precisely when G is a reflection group, as illustrated in Example 5.2.

Theorem 5.3 (Sheppard-Todd-Chevalley) *Let G be a finite group acting on a polynomial ring $R = \mathbb{C}[x_1, \ldots, x_n]$. Then R^G is a polynomial ring (and $D(R^G)$ is a Weyl algebra) if and only if G is a reflection group.*

Kantor [15, Theorem 4 in section 3.3.1] showed that the other extreme case – when G contains no pseudoreflections – characterizes the case where π_* is surjective.

Theorem 5.4 (Kantor) *When G is a finite group acting on a polynomial ring $R = \mathbb{C}[x_1, \ldots, x_n]$, the map π_* is a surjection precisely when G contains no pseudoreflections. In this case, $D(R^G) = D(R)^G$.*

When G acting on a polynomial ring R contains some pseudoreflections, but G is not a reflection group, we factor the action of G on $X = \mathrm{Spec}(R)$ as follows. First note that the subgroup P generated by the pseudoreflections is a normal subgroup of G. To see this, it is enough to check that if p is a pseudoreflection and $g \in G$, then $gpg^{-1} \in P$; this follows since $(g \cdot X^p) \subset X^{gpg^{-1}}$ so $\mathrm{codim}(X^{gpg^{-1}}) \leq \mathrm{codim}(g \cdot X^p) = 1$. Now G/P acts[1] on the polynomial ring R^P and $R^G = (R^P)^{G/P}$.

Since G/P contains no pseudoreflections, the map $\pi_* : D(R^P)^{G/P} \to D(R^G)$ is an isomorphism, so $D(R^G)$ can be described as the ring of invariant differential operators of the group G/P acting on the Weyl algebra $D(R^P)$. It is in this sense that we will be able to describe $D(R^G)$ for finite groups G.

Example 5.5 We return to Example 3.2 and compute a presentation for $D(R^G)$ in terms of generators and relations. First note that the subgroup $P \triangleleft G$ of pseudoreflections is generated by γ, the reflection in the x_2x_3-plane. Direct observation shows that $R^P = \mathbb{C}[x_1^2, x_2, x_3]$. Write $z = x_1^2$. The quotient G/P is generated by the image of δ and this element acts on R^P by sending z to itself, x_2 to x_3, and x_3 to $-x_2$. Applying Derksen's algorithm to G/P acting on $D(R^P) = \mathbb{C}\langle z, x_2, x_3, \partial_z, \partial_2, \partial_3 \rangle$ gives six generators for $D(R^P)^{G/P} = D(R^G)$:

$$\left\{ \partial_z, z, \partial_2^2 + \partial_3^2, x_3\partial_2 - x_2\partial_3, x_2\partial_2 + x_3\partial_3, x_2^2 + x_3^2 \right\}.$$

[1] I'm grateful to Gregor Kemper who provided the following short proof that the action of G/P on R^P is linear in the non-modular case. The vector space $(R^P_{>0})^2$ has a G/P-complement U with basis B. Then G/P acts by linear transformations on the vectors of B. But by the homogeneous version of Nakayama's lemma, B generates R^P minimally.

Let a, b, c, d, e, f denote these six operators and let $[a], [b], [c], [d], [e], [f]$ denote their symbols. The symbols of these six operators generate $Gr(D(R^P))^{G/P}$ and an elimination computation shows that there is only one syzygy on these generators, $[d]^2 + [e]^2 - [c][f] = 0$. This lifts to a single syzygy on the generators of $D(R^P)^{G/P} = D(R^G)$,

$$d^2 + e^2 - cf + 4e + 4 = 0. \tag{5.1}$$

There are also four nontrivial commutator relations among the generators:

$$[a, b] = 1, \quad [c, e] = 2c, \quad [c, f] = 4e + 4, \quad [e, f] = 2f.$$

The third commutator relation shows that the syzygy (5.1) has a nicer form,

$$d^2 + e^2 + fc = 0 \tag{5.2}$$

The commutator relations, together with the syzygy (5.2) generate the two-sided ideal of relations in among the generators of $D(R^G)$.

6 Differential Operators on G(2,4)

We now give an example involving $G = SL_2\mathbb{C}$. If V is a 2-dimensional complex vector space, then $\mathbb{C}[V^4]^{SL_2\mathbb{C}}$ is the coordinate ring of the affine cone over the Grassmannian $G(2, 4)$ of 2-planes in $\mathbb{C}^4$. Let $\{x_{1i}, x_{2i}\}$ be coordinate functions on the i^{th} copy of V in V^4, then the Fundamental Theorems of Invariant Theory for SL_n (see DK, Theorems 4.4.4 and 4.4.5) imply that $\mathbb{C}[V^4]^{SL_2\mathbb{C}}$ is generated by six polynomials $[12], [13], [14], [23], [24], [34]$, where $[ij] = x_{1i}x_{2j} - x_{1j}x_{2i}$ is the 2×2 minor of the matrix

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix}.$$

The ideal of relations on these generators is generated by the Plücker relation

$$[12][34] - [13][24] + [14][23] = 0.$$

We apply Derksen's algorithm to compute the ring of differential operators on the affine cone over the Grassmannian $G(2, 4)$.

Example 6.1 We represent the group $SL_2\mathbb{C}$ as the vanishing set of $a_1a_3 - a_2a_4 - 1$, where the point (a_1, a_2, a_3, a_4) corresponds to the matrix $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in SL_2\mathbb{C}$. The group G acts on the x_{ij} by matrix multiplication on the left. This induces an action on $GrD(\mathbb{C}[V^4])$; writing ξ_{ij} for the symbol of $\partial/\partial x_{ij}$, the matrix corresponding to

(a_1, a_2, a_3, a_4) acts on

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ \xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} \\ \xi_{21} & \xi_{22} & \xi_{23} & \xi_{24} \end{bmatrix} \quad \text{to give} \quad \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ a_3 & a_4 & 0 & 0 \\ 0 & 0 & a_4 & -a_3 \\ 0 & 0 & -a_2 & a_1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ \xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} \\ \xi_{21} & \xi_{22} & \xi_{23} & \xi_{24} \end{bmatrix}.$$

Applying Derksen's algorithm we obtain 28 generators for $(GrD(\mathbb{C}[V^4]))^{SL_2\mathbb{C}}$. Each of these is already invariant under $SL_2\mathbb{C}$, so there is no need to apply the Reynolds operator. Lifting these operators gives generators for $D(\mathbb{C}[V])^{SL_2\mathbb{C}}$:

$$\left\{ \begin{array}{llll} \partial_{14}\partial_{23} - \partial_{13}\partial_{24}, & \partial_{14}\partial_{22} - \partial_{12}\partial_{24}, & \partial_{13}\partial_{22} - \partial_{12}\partial_{23}, & \partial_{14}\partial_{21} - \partial_{11}\partial_{24}, \\ \partial_{13}\partial_{21} - \partial_{11}\partial_{23}, & \partial_{12}\partial_{21} - \partial_{11}\partial_{22}, & x_{14}\partial_{14} + x_{24}\partial_{24}, & x_{13}\partial_{14} + x_{23}\partial_{24}, \\ x_{12}\partial_{14} + x_{22}\partial_{24}, & x_{11}\partial_{14} + x_{21}\partial_{24}, & x_{14}\partial_{13} + x_{24}\partial_{23}, & x_{13}\partial_{13} + x_{23}\partial_{23}, \\ x_{12}\partial_{13} + x_{22}\partial_{23}, & x_{11}\partial_{13} + x_{21}\partial_{23}, & x_{14}\partial_{12} + x_{24}\partial_{22}, & x_{13}\partial_{12} + x_{23}\partial_{22}, \\ x_{12}\partial_{12} + x_{22}\partial_{22}, & x_{11}\partial_{12} + x_{21}\partial_{22}, & x_{14}\partial_{11} + x_{24}\partial_{21}, & x_{13}\partial_{11} + x_{23}\partial_{21}, \\ x_{12}\partial_{11} + x_{22}\partial_{21}, & x_{11}\partial_{11} + x_{21}\partial_{21}, & x_{14}x_{23} - x_{13}x_{24}, & x_{14}x_{22} - x_{12}x_{24}, \\ x_{13}x_{22} - x_{12}x_{23}, & x_{14}x_{21} - x_{11}x_{24}, & x_{13}x_{21} - x_{11}x_{23}, & x_{12}x_{21} - x_{11}x_{22} \end{array} \right\}.$$

Furthermore, in an important paper about the behavior of π_* [31] Gerald Schwarz showed that the LS-alternative holds for $SL_2\mathbb{C}$: either $\mathbb{C}[V^4]^{SL_2\mathbb{C}}$ is regular or the map $\pi_* : D(\mathbb{C}[V^4])^{SL_2\mathbb{C}} \to D(\mathbb{C}[V^4]^{SL_2\mathbb{C}})$ is surjective. Since $\mathbb{C}[V]^{SL_2\mathbb{C}}$ represents a cone it is not a regular ring so π_* is surjective. It follows that the generators for $D(\mathbb{C}[V^4])^{SL_2\mathbb{C}}$ generate $D(\mathbb{C}[V^4]^{SL_2\mathbb{C}})$, when restricted to $\mathbb{C}[V^4]^{SL_2\mathbb{C}}$.

This example illustrates the power and the generality of the Gröbner basis techniques, but the result also follows from the Fundamental Theorems of Invariant Theory for $SL_n\mathbb{C}$ (for details see see [29, sections 9.3 and 9.4]). We now explain this connection.

Let V be an n-dimensional complex vector space and let V^* be the dual space of V. Then $\mathbb{C}[V^r \oplus (V^*)^s]$ is generated by the coordinates x_{ij} and ξ_{ij} ($1 \leq i \leq r, 1 \leq j \leq r$; here $\xi_{ij} = \mathbf{x}^*_{ij}$). If $\langle \cdot, \cdot \rangle : V \times V^* \to \mathbb{C}$ is the canonical pairing, for each $i \leq r$ and $j \leq s$ we have an invariant $\langle ij \rangle : V^r \oplus (V^*)^s \to \mathbb{C}$ that sends $(v_1, \ldots, v_r, w_1, \ldots, w_s)$ to $\langle v_i, w_j \rangle$. In coordinates $\langle ij \rangle = \sum_{k=1}^n x_{ki}\xi_{kj}$.

There are other invariants too. If $1 \leq i_1 < i_2 < \cdots < i_n \leq r$, we have a bracket invariant $[i_1 i_2 \cdots i_n] : V^r \oplus (V^*)^s \to \mathbb{C}$ given by

$$(v_1, \ldots, v_r, w_1, \ldots, w_s) \to \det(v_{i_1} v_{i_2} \cdots v_{i_n}).$$

This is an operator of degree n that only involves the x_{ij}. As well, if $1 \leq j_1 < j_2 < \cdots < j_n \leq s$, we have an invariant $|j_1 j_2 \cdots j_n| : V^r \oplus (V^*)^s \to \mathbb{C}$ given by

$$(v_1, \ldots, v_r, w_1, \ldots, w_s) \to \det(w_{j_1} w_{j_2} \cdots w_{j_n}).$$

This is an operator of total degree n that only involves the ξ_{ij}.

Theorem 6.2 (Fundamental Theorem of Invariant Theory for $\mathbf{SL_n}\mathbb{C}$) *Let V be an n-dimensional complex vector space. The invariant ring*

$$\mathbb{C}[V^r \oplus (V^*)^s]^{SL_n\mathbb{C}}$$

is generated by all $\langle ij\rangle$ ($1 \leq i \leq r, 1 \leq j \leq s$), all $[i_1 i_2 \cdots i_n]$ ($1 \leq i_1 < i_2 < \cdots < i_n \leq r$) and all $|j_1 j_2 \cdots j_n|$ ($1 \leq j_1 < j_2 < \cdots < j_n \leq s$). The relations among these generators are of five types:
(a) For $1 \leq i_1 < i_2 < \cdots < i_n \leq r$ and $1 \leq j_1 < j_2 < \cdots < j_n \leq s$:
det$(\langle i_k j_\ell\rangle)_{k,\ell=1}^n = [i_1 i_2 \cdots i_n]|j_1 j_2 \ldots j_n|$
(b) For $1 \leq i_1 < i_2 < \cdots < i_{n+1} \leq r$ and $1 \leq j \leq s$:
$\sum_{k=1}^{n+1}(-1)^{k-1}[i_1 i_2 \cdots \hat{i_k} \cdots i_{n+1}]\langle i_k j\rangle = 0$
(c) For $1 \leq j_1 < j_2 < \cdots < j_{n+1} \leq s$ and $1 \leq i \leq r$:
$\sum_{k=1}^{n+1}(-1)^{k-1}\langle i j_k\rangle|j_1 j_2 \cdots \hat{j_k} \cdots j_{n+1}| = 0$
(d) For $1 \leq i_1 < i_2 < \cdots < i_{n-1} \leq r$ and $1 \leq j_1 < j_2 < \cdots < j_{n+1} \leq r$:
$\sum_{k=1}^{n+1}(-1)^{k-1}[i_1 i_2 \cdots i_{n-1} j_k][j_1 j_2 \cdots \hat{j_k} \cdots j_{n+1}]$
(e) For $1 \leq i_1 < i_2 < \cdots < i_{n-1} \leq s$ and $1 \leq j_1 < j_2 < \cdots < j_{n+1} \leq s$:
$\sum_{k=1}^{n+1}(-1)^{k-1}|i_1 i_2 \cdots i_{n-1} j_k|\ |j_1 j_2 \cdots \hat{j_k} \cdots j_{n+1}|$.

Now $(GrD(\mathbb{C}[V^4]))^{SL_2\mathbb{C}} = \mathbb{C}[V^4 \oplus (V^*)^4]^{SL_2\mathbb{C}}$ so we can apply Theorem 6.2 in the case $r = s = 4$. We see that $(GrD(\mathbb{C}[V^4]))^{SL_2\mathbb{C}}$ is generated by twenty eight operators: the six $[ij]$, the six $|ij|$ and the sixteen $\langle ij\rangle$. These are precisely the operators found in Example 6.1.

There are 156 relations among the generators of $(GrD(\mathbb{C}[V^4])^{SL_2\mathbb{C}}$, 36 each of types (a), (d) and (e) and 24 each of types (b) and (c). Each of these extends to an ordered relation on $D(\mathbb{C}[V^4]^{SL_2\mathbb{C}}) = (GrD(\mathbb{C}[V^4])^{SL_2\mathbb{C}}$. In most cases no modification of the formula is needed, if we take care to write the relations in the order given by Theorem 6.2. However, the relations in part (a) need to be properly interpreted. We explain how to do this for the case $SL_2\mathbb{C}$. Each term in the determinant $\det(\langle i_k j_\ell\rangle)_{k,\ell=1}^n$ involves the product of two terms $\langle i_k j_\ell\rangle$. When possible we write these products in an order where the last entry of the first term does not coincide with the first entry of the second term. If this can be achieved, then no modification to the formula in part (a) is necessary. If not, then we have a term $\langle ab\rangle\langle ba\rangle$ in the expansion of the determinant and to compensate we must add $\langle aa\rangle$ to the right-hand side of the relation:

$$\det\begin{pmatrix}\langle aa\rangle & \langle ab\rangle \\ \langle ba\rangle & \langle bb\rangle\end{pmatrix} + \langle aa\rangle = \langle aa\rangle\langle bb\rangle - \langle ab\rangle\langle ba\rangle + \langle aa\rangle = [ab]|ab|.$$

The commutator relations among the 28 generators also give rise to relations. Unfortunately, many of these are non-trivial, $[a_i, a_j] \neq 0$ in 156 of 406 cases. However, we do have a compact description of the commutator relations:

$$\begin{aligned}
&[|ij|, [ij]] = \langle ii\rangle + \langle jj\rangle + 2,\\
&[|ij|, [ik]] = \langle kj\rangle,\\
&[[ij], \langle ii\rangle] = [ji],\\
&[|ij|, \langle ii\rangle] = |ij|,\\
&[[ij], \langle kj\rangle] = [ki],\\
&[|ij|, \langle ik\rangle] = |kj|,\\
&[\langle ij\rangle, \langle ji\rangle] = \langle ii\rangle - \langle jj\rangle,\\
&[\langle ij\rangle, \langle ii\rangle] = -\langle ij\rangle,\\
&[\langle ij\rangle, \langle jj\rangle] = \langle ij\rangle,\\
&[\langle ij\rangle, \langle ki\rangle] = -\langle kj\rangle,\\
&[\langle ij\rangle, \langle jk\rangle] = \langle ik\rangle.
\end{aligned}$$

The relations described so far are enough to determine $D(R)^G$. However, the map $\pi_* : D(R)^G \to D(R^G)$ is not injective. It is known [31] that the kernel of π_* consists of the G-stable part of the left ideal of $D(R)$ generated by the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2\mathbb{C}$. The Lie algebra $\mathfrak{sl}_2$ is generated by three elements g_{12}, g_{21} and $g_{11} - g_{22}$, where g_{ij} corresponds to the adjoint action of the matrix E_{ij} with a 1 in the $(i, j)^{\text{th}}$ position and zero elsewhere. Explicitly,

$$\begin{aligned}
g_{12} &= x_{11}\partial_{21} + x_{12}\partial_{22} + x_{13}\partial_{23} + x_{14}\partial_{24},\\
g_{21} &= x_{21}\partial_{11} + x_{22}\partial_{12} + x_{23}\partial_{13} + x_{24}\partial_{14},\\
g_{11} - g_{22} &= x_{11}\partial_{11} + x_{12}\partial_{12} + x_{13}\partial_{13} + x_{14}\partial_{14}\\
&\quad -x_{21}\partial_{21} - x_{22}\partial_{22} - x_{32}\partial_{32} - x_{42}\partial_{42}.
\end{aligned}$$

We can compute the part of the left ideal generated by g_{11}, g_{22} and $g_{11} - g_{22}$ that is G-invariant by intersecting with the subalgebra generated by the invariants. This is a gigantic computation that was performed in SINGULAR using the nctools package. In an extension of $GrD(R)$, $\mathbb{C}[x_{ij}, \xi_{ij}, [ij], |ij|, \langle ij\rangle]$, we form an ideal containing $g_{12}, g_{21}, g_{11} - g_{22}$, and the relations that describe $[ij], |ij|$, and $\langle ij\rangle$ in terms of the x_{ij} and ξ_{ij}. Imposing the block order that places the x_{ij} and ξ_{ij} in the first block and the $[ij], |ij|$ and $\langle ij\rangle$ in the second block, we compute a Gröbner basis of the ideal. After intersecting with $\mathbb{C}[[ij], |ij|, \langle ij\rangle]$ we have 191 polynomials in the Gröbner basis. These polynomials generate the graded kernel K of the map $\pi_* : GrD(R)^G \to GrD(R^G)$. Each of these graded generators extend to an element in $D(R)^G$.

As a result, we've shown that the ring of differential operators on the Grassmannian $D(G(2, 4)) = D(R^G)$ is generated by 28 operators satisfying a two-sided ideal of relations generated by the commutator relations and the extensions of the relations from K.

Among the extensions of the generators of K is the interesting element

$$\theta(\theta + 2) - 4\sum_{i<j}[ij]|ij|, \tag{6.1}$$

where θ is the operator

$$\langle 11\rangle + \langle 22\rangle + \langle 33\rangle + \langle 44\rangle.$$

The generator (6.1) is a multiple of the Casimir operator of $\mathfrak{sl}_2\mathbb{C}$. This is easily verified by an explicit computation as follows. The Lie algebra $\mathfrak{sl}_2\mathbb{C}$ has inner product given by the Killing form

$$\kappa(\delta, \gamma) = \mathrm{Tr}(\mathrm{ad}(\delta), \mathrm{ad}(\gamma)),$$

where Tr is the trace and $\mathrm{ad}(\delta)(\gamma) = \delta\gamma - \gamma\delta$ is the adjoint action of $\delta \in \mathfrak{sl}_2\mathbb{C}$ on itself. A dual basis for $\mathfrak{sl}_2\mathbb{C}$ with respect to the Killing form is given by $g_{12}/4, g_{21}/4,$ and $(g_{11} - g_{22})/8$. Then the Casimir operator [8, Definition 4.5.10] is just given by

$$\frac{g_{12}g_{21}}{4} + \frac{g_{21}g_{12}}{4} + \frac{(g_{11} - g_{22})^2}{8}.$$

Explicit computation in $D(R)$ then shows that the Casimir operator is equal to the operator (6.1) divided by 8.

At one time I conjectured that the kernel of π_* was a two-sided ideal of $D(R)^G$ generated by the Casimir operator. Many people suggested that this should be the case since the Casimir operator generates the center of $\mathfrak{sl}_2\mathbb{C}$; however, it turns out that the Casimir operator does not generate the kernel of π_* (see Traves [35] for details).

7 Conclusion

This paper dealt with constructive techniques in invariant theory for rings of differential operators. Derksen's algorithm was applied to $GrD(R)$ in order to compute $Gr(D(R)^G)$ and then the relationship between $D(R)^G$ and $D(R^G)$ was used to find generators and relations for the ring of differential operators on the quotient variety, $D(R^G)$. In particular, the generators and relations for $D(G(2, 4))$ were described.

Levasseur and Stafford [18] work out many other cases of invariant rings of differential operators for the classical groups. As well, Schwarz's work on lifting differential operators [31] is crucial in understanding the relation between $D(R)^G$ and $D(R^G)$.

The ring of invariants R^G is a module over the invariant differential operators $D(R)^G$. Of course, in many cases R^G is a simple $D(R)^G$ module, but if we restrict ourselves to looking at R^G as a module over a subalgebra of $D(R)^G$, then it may well be possible to find many fewer module generators for R^G. This topic is central to invariant theory in prime characteristic, where the subalgebra of choice is the Steenrod algebra (see Smith [33] for details). Pleskin and Robertz [28] investigate the characteristic zero case, but one gets the feeling that much more can be said about the theory of invariant rings R^G as modules over appropriately chosen submodules of $D(R)^G$.

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