

## NAKAI'S CONJECTURE FOR VARIETIES SMOOTHED BY NORMALIZATION

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ABSTRACT. The notion of D-simplicity is used to give a short proof that varieties whose normalization is smooth satisfy Ishibashi's extension of Nakai's conjecture to arbitrary characteristic. This gives a new proof of Nakai's conjecture for curves and Stanley-Reisner rings.

### INTRODUCTION

Nakai's conjecture concerns a very natural question: can differential operators detect singularities on algebraic varieties? On a smooth complex variety, it is well known that the ring of differential operators is generated by derivations. Nakai asked whether the converse holds: if the ring of differential operators is generated by derivations, is the variety smooth? In this paper, the notion of D-simplicity is used to give a short proof that varieties whose normalization is smooth satisfy Ishibashi's extension [2] of Nakai's conjecture to arbitrary characteristic. For example, any variety whose irreducible components are smooth satisfies Nakai's conjecture, giving a proof that Nakai's conjecture holds for Stanley-Reisner rings. Furthermore, this gives a simple new characteristic-independent proof of Nakai's conjecture for curves. The argument is quite short and rederives characteristic-dependent results of Mount and Villamayor (characteristic zero) [4] and Ishibashi (prime characteristic) [2].

### DEFINITIONS AND NOTATION

Let  $X = \text{Spec}(R)$  be an affine algebraic variety defined over a field  $k$  of characteristic zero. When  $R$  is regular (that is, when  $X$  is smooth), the ring of differential operators  $D(R/k)$  (see *EGA* [1] or [3]) equals  $\text{der}(R/k)$ , the  $R$ -subalgebra generated by the derivations (see McConnell and Robson [3, Corollary 15.5.6]). Nakai conjectured that this condition characterizes nonsingularity:  $R$  is regular if and only if  $\text{der}(R/k) = D(R/k)$ . Ishibashi [2] extended Nakai's conjecture to varieties defined over an arbitrary perfect field.

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A Hasse-Schmidt derivation  $\Delta = \{\delta_n\}_{n=0}^\infty \subseteq \text{End}_k(R)$  is a collection of  $k$ -linear endomorphisms of  $R$  such that  $\delta_0 = \text{id}_R$  and

$$\delta_n(ab) = \sum_{i+j=n} \delta_i(a)\delta_j(b).$$

For example, if  $R$  has characteristic zero and  $d$  is a derivation,  $\delta_n = \frac{1}{n!}d^n$  determines a Hasse-Schmidt derivation. Let  $HS(R/k)$  be the  $R$ -algebra generated by the components  $\delta_n$  of Hasse-Schmidt derivations on  $R$ .  $HS(R/k)$  is a subalgebra of  $D(R/k)$  and if  $R$  has characteristic zero,  $\text{der}(R/k) = HS(R/k)$ . In characteristic zero Grothendieck [1, 16.11.2 and 17.12.4] showed that when  $R$  is smooth over  $k$ ,  $HS(R/k) = D(R/k)$ ; a proof in arbitrary characteristic can be found in Traves [9]. Ishibashi's extension<sup>1</sup> of Nakai's conjecture is that  $R$  is smooth over  $k$  if and only if  $HS(R/k) = D(R/k)$ . The main result of this paper is that varieties whose normalization is smooth satisfy this extension of Nakai's conjecture.

#### PRELIMINARY RESULTS

Let  $R$  be a reduced algebra of finite type over a perfect field  $k$ . For a reduced ring  $R$ , the normalization  $R'$  of  $R$  is the integral closure of  $R$  in its total ring of quotients,  $L = S^{-1}R$ , where  $S$  is the multiplicative set of nonzerodivisors in  $R$ . The conductor of  $R'$  into  $R$  is

$$C = \{c \in R : cR' \subseteq R\}.$$

The conductor is an ideal of both  $R$  and  $R'$ .

**Lemma 1.** (1) *The conductor is  $HS(R/k)$ -stable.*

(2) *Powers of  $HS(R/k)$ -stable ideals are  $HS(R/k)$ -stable.*

*Proof.* Part (1) is well-known: see Seidenberg [6, Corollary on page 169 and section 5]. To establish (2) it suffices to show that for  $a_1, \dots, a_s$  in the  $HS(R/k)$ -stable ideal  $I$  and  $\Delta = \{\delta_n\}$  a Hasse-Schmidt derivation,  $\delta_n(a_1 \cdots a_s) \in I^s$ . Now

$$\delta_n(a_1 \cdots a_s) = \sum_{i+j=n} \delta_i(a_1)\delta_j(a_2 \cdots a_s)$$

and the claim follows by induction on  $s$ . □

**Lemma 2.** *The conductor is not contained in any minimal prime of  $R$ .*

*Proof.* The conductor  $C$  equals  $\text{Ann}_R(\frac{R'}{R})$ . If  $C \subseteq P$  with  $P$  a minimal prime of  $R$ , then, since  $\text{Supp}(\frac{R'}{R}) = \mathbb{V}(\text{Ann}(\frac{R'}{R}))$ ,  $P \in \text{Supp}(\frac{R'}{R})$ . Localizing the exact sequence of  $R$ -modules

$$0 \rightarrow R \rightarrow R' \rightarrow \frac{R'}{R} \rightarrow 0$$

at the minimal prime  $P$  gives an exact sequence

$$0 \rightarrow R_P \rightarrow R'_P \rightarrow (\frac{R'}{R})_P \rightarrow 0.$$

But since  $P$  is a minimal prime of a reduced ring,  $R_P$  is a field and the normalization map  $R_P \rightarrow R'_P$  is an isomorphism. This forces  $(\frac{R'}{R})_P = 0$ , a contradiction. So  $C$  is not contained in any minimal prime of  $R$ . □

<sup>1</sup>Actually, Ishibashi requires that  $k$  be algebraically closed and conjectures  $R$  regular, but it is clear that smoothness is the relevant notion

**Lemma 3.** *If  $R$  is a domain that is smooth over  $k$ , then  $R$  is  $D(R/k)$ -simple.*

*Proof.* Since  $k$  is perfect,  $R$  is regular. In the characteristic zero case, it is well known that  $R$  is  $D(R/k)$ -simple (see McConnell and Robson [3, Theorem 15.3.8 and Corollary 15.5.6]). For the prime characteristic case, note that since  $R$  is regular,  $R$  is strongly F-regular and since  $R$  is an algebra of finite type over a perfect field,  $R$  is F-finite. Smith [7, Theorem 2.2] has shown that strongly F-regular F-finite domains are  $D(R/k)$ -simple.  $\square$

1. NAKAI'S CONJECTURE

**Theorem 4.** *Let  $R$  be a reduced  $k$ -algebra of finite type and let  $R'$  be its integral closure in its total ring of quotients. If  $R'$  is a product of  $D$ -simple rings and  $HS(R/k) = D(R/k)$ , then  $R$  is normal.*

*Proof.* Note that  $R'$  is isomorphic to  $R_1 \times \cdots \times R_t$ , where  $R_i$  is the normalization of  $\frac{R}{P_i}$  with the  $\{P_i\}$  ranging over the minimal primes of  $R$ . By Lemma 1 (1),  $C$  is  $HS(R/k)$ -stable and by Lemma 1 (2),  $C^2$  is also  $HS(R/k)$ -stable. Thus,  $C^2$  is  $D(R/k)$ -stable.

Assume that  $C^2 \neq C$ . Since  $C$  is not contained in any of the minimal primes  $P_i$  of  $R$ , there are elements  $c_i \in C \setminus P_i$  with some  $c_j \notin C^2$ . To see this, take  $x \in C \setminus C^2$  and note that  $x \neq 0 \Rightarrow x \notin \bigcap P_i \Rightarrow x$  is not in some  $P_j$ . Set  $c_j = x$  and pick the other  $c_i \in C \setminus P_i$ . Let  $c = (c_1, \dots, c_t) \in R'$ , after identifying  $R'$  with the product in the first paragraph. Then  $c \in C \setminus C^2$  and  $c$  is nonzero in each component. By the  $D$ -simplicity of each of the  $R_i$ , there is an operator  $\theta = (\theta_1, \dots, \theta_t) \in D(R_1) \times \cdots \times D(R_t)$ , such that  $\theta(c^2) = 1$ . If each  $\theta_i \in D(R_i)$  is an operator of order  $\leq n_i$  then  $\theta = (\theta_1, \dots, \theta_t)$  maps  $R'$  to itself and  $\theta$  is a differential operator of order  $\leq n = \max(n_i)$ . Thus,  $\theta \in D(R')$  and  $c\theta \in D(R)$ . Now  $(c\theta)(c^2) = c \notin C^2$ , contradicting the fact that  $C^2$  is  $D(R/k)$ -stable. This forces  $C^2 = C$ .

In fact,  $C = R$ . Indeed, if  $C$  is contained in a maximal ideal  $m$ , then  $C_m = C_m^2 \subset mC_m \subset C_m$ , so  $mC_m = C_m$ . Now Nakayama's lemma forces  $C_m = 0$ . But then  $C$  must consist of zero divisors, contradicting Lemma 2. So  $C = R$  and  $R$  is normal.  $\square$

**Theorem 5.** *If  $HS(R/k) = D(R/k)$  and the normalization  $R'$  of  $R$  is smooth over  $k$ , then  $R$  is smooth over  $k$ .*

*Proof.* Since the normalization  $R'$  of  $R$  is a product of smooth domains,  $R'$  is a product of  $D$ -simple rings (by Lemma 3). The result now follows from Theorem 4.  $\square$

This theorem says that the ring of differential operators of a singular complex variety whose normalization is smooth is not generated by derivations. Thus,  $D(R/\mathbb{C})$  is complicated even for very mild singularities (those that can be resolved by normalization). We expect that  $D(R/\mathbb{C})$  will become more complicated as the singularities become worse. This provides further evidence for Nakai's conjecture.

The theorem shows that Nakai's conjecture holds for reduced varieties smoothed by normalization; for example, curves.

**Corollary 6.** *Let  $R$  be a reduced  $k$ -algebra of finite type, where  $k$  is a perfect field. If  $R$  is 1-dimensional and  $HS(R/k) = D(R/k)$ , then  $R$  is smooth over  $k$ . In*

particular, if  $R$  is a domain of characteristic 0 and  $\text{der}(R/k) = D(R/k)$ , then  $R$  is regular.

Theorem 5 can also be used to recover a result due to Schreiner [5] (also, see Traves [8]): Nakai's conjecture holds for Stanley-Reisner rings (that is, for subvarieties of  $\mathbb{A}_k^N$  which are the union of coordinate subspaces). More generally, the theorem implies that Nakai's conjecture holds for varieties all of whose components are smooth, as remarked by Lazarsfeld.

**Corollary 7.** *Let  $R$  be a reduced  $k$ -algebra of finite type, where  $k$  is a perfect field. If  $HS(R/k) = D(R/k)$  and if  $\frac{R}{P}$  is smooth over  $k$  for each minimal prime  $P$  of  $R$ , then  $R$  is smooth over  $k$ . In particular, if  $R$  is a Stanley-Reisner ring and  $HS(R/k) = D(R/k)$ , then  $R$  is a polynomial ring.*

Both Corollary 6 and Corollary 7 follow immediately from Theorem 5: just observe that the normalization of  $X = \text{Spec}(R)$  is isomorphic to the disjoint union of the normalization of the components of  $X$ , each of which is smooth over  $k$ .

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