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Note

The reconstruction conjecture and edge ideals

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Abstract

Given a simple graph G on n vertices, we prove that it is possible to reconstruct several algebraic properties of the edge ideal from the deck of G, that is, from the collection of subgraphs obtained by removing a vertex from G. These properties include the Krull dimension, the Hilbert function, and all the graded Betti numbers $\beta_{i,j}$ where $j<n$. We also state many further questions that arise from our study.

MSC: 05C60; 13F55

Keywords: Reconstruction conjecture; Edge ideals; Hilbert function; Graded Betti number

1. Introduction

The graph reconstruction conjecture, posed by Kelly and Ulam in 1941 (see [1]), says that every simple graph G on $n \geqslant 3$ vertices is determined by the collection of subgraphs obtained, up to isomorphism, by deleting one vertex from G; this collection is called the *deck* of G, and each vertex deleted subgraph is called a *card*. This conjecture has proved notoriously difficult, and has motivated a large amount of work in graph theory. It is known, for example, that trees are reconstructible, as are all graphs of up to 11 vertices, and all disconnected graphs. It is also known that almost all graphs are reconstructible from three carefully chosen cards from their deck. Bondy [1] summarizes the work done on this problem.

In recent years, much work has been done on studying graphs from an algebraic point of view. The *edge ideal* of a simple graph is the ideal generated by certain degree two monomials, where each monomial is the product of the two vertices joined by an edge of the graph. Edge ideals were introduced by Villarreal [9,10], who used the combinatorial properties of graphs to produce algebraic statements. In particular, Villarreal and his coauthors studied Rees rings and Cohen-Macaulay properties of edge ideals [9,10,8]. Hà, Roth and Van Tuyl [4,7], among others, have studied resolutions of edge ideals, finding recursive methods to compute Betti numbers for special classes of graphs.

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Our goal in this paper is to consider the reconstruction of algebraic invariants associated to the edge ideal of a simple graph. An invariant of a graph is said to be *reconstructible* if it can be determined from its deck. For example, let G be a graph on n vertices in $V(G)$ and edge set $E(G)$. The number of edges of G is reconstructible because $(n-2)\#E(G)$ is the total number of edges of all the cards of the deck. As well, the degree sequence of the graph G (this is the non-increasing sequence of degrees of all the vertices of G) is reconstructible since it equals a rearrangement of the elements of the set

$$\{\#E(G) - \#E(C) : C \text{ a card of the deck of } G\}.$$

Similarly, a property of a graph is said to be *recognizable* if one can determine whether this property holds for a graph G just by looking at the deck of G.

The same principle can be applied to algebraic invariants of the graph, that is, invariants of the quotient S of a polynomial ring by the edge ideal. The main content of this paper is to prove that various algebraic invariants of the graph are reconstructible. Most notably we will prove that the multiplicity, dimension and Hilbert function of S, and the graded Betti numbers of S in non-maximal degrees are reconstructible.

Just as the reconstruction conjecture has proven to be extremely complicated despite its simple statement, we have found that several of the reconstructible algebraic properties are either straightforward to verify, or seem to be very difficult. For this reason, we have included many questions and examples in the manuscript, which we hope will inspire further investigations on this beautiful topic.

2. Preliminaries

We begin by reviewing some basic facts about edge ideals and Stanley–Reisner ideals. Villarreal's book [10] is a comprehensive introduction to these topics.

Definition 2.1 (*Simplicial complexes and graphs*). A *simplicial complex* Δ over a set of vertices $V=V(\Delta)=\{x_1, \ldots, x_n\}$ is a collection of subsets of V, with the property that $\{x_i\} \in \Delta$ for all i, and if $F \in \Delta$ then all subsets of F (including the empty set) are also in Δ. An element of Δ is called a *face* of Δ, and the *dimension* of a face F of Δ is defined to be $|F| - 1$, where $|F|$ is the number of vertices of F. The faces of dimensions 0 and 1 are called *vertices* and *edges*, respectively, and $\dim \emptyset = -1$. The maximal faces of Δ under inclusion are called *facets* of Δ. The dimension of the simplicial complex Δ is the maximal dimension of its facets.

The f-*vector* of Δ is the integer vector $(f_0, f_1, \ldots, f_d)$, with f_i the number of faces of dimension i in Δ.

A (simple) *graph* G is a simplicial complex of dimension 1. We denote the set of vertices and the set of edges of G by $V(G)$ and $E(G)$, respectively.

All graphs that we consider in this paper are simple graphs.

A *vertex cover* for a graph G is a subset A of $V(G)$ that intersects every edge of G. If A is a minimal element (under inclusion) of the set of vertex covers of G, it is called a *minimal vertex cover*. A graph G is *unmixed* if all of its minimal vertex covers have the same cardinality.

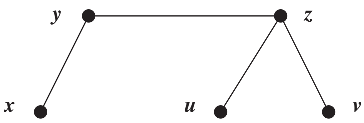
Definition 2.2 (*Stanley–Reisner ideal*). Let Δ be a simplicial complex over n vertices $x_1, \ldots, x_n$. Let $R=k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. The *Stanley–Reisner ideal* of Δ is the ideal I_Δ of R generated by the square-free monomials $x_{i_1} \ldots x_{i_r}$, where $\{x_{i_1}, \ldots, x_{i_r}\}$ is not a face of Δ. Similarly, if I is a square-free monomial ideal in R then the unique simplicial complex Δ on n vertices with Stanley–Reisner ideal I is the Stanley–Reisner complex of I.

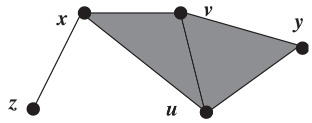
Definition 2.3 (*Edge ideal*). Let G be a graph with n vertices $x_1, \ldots, x_n$. Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. The *edge ideal* $I(G)$ of G is the ideal of R generated by the square-free monomials $x_i x_j$, where $\{x_i, x_j\}$ is an edge of G.

Clearly, a graph over a given set of vertices is uniquely determined by its edge ideal. The *Stanley–Reisner complex* Δ_G of a graph G is the Stanley–Reisner complex of the edge ideal $I(G)$.

Definition 2.4 (*Induced subgraph*). Given a subset B of vertices of a graph G, the subgraph G_B induced by B is the graph with vertex set B and edges $\{\{x, y\} \in E(G) : x, y \in B\}$.

Example 2.5. Let $I(G)=(xy, yz, zu, zv)$ be the edge ideal of the graph G (pictured on the left) in the ring $k[x, y, z, u, v]$. The Stanley–Reisner complex of $I(G)$ is Δ_G, pictured below on the right.



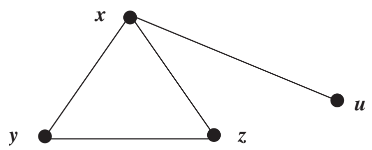


Note that the facets of Δ_G correspond to maximal subsets of $B \subseteq \{x, y, z, u, v\}$ where the induced graph G_B has no edges (i.e. G_B consists of only isolated vertices).

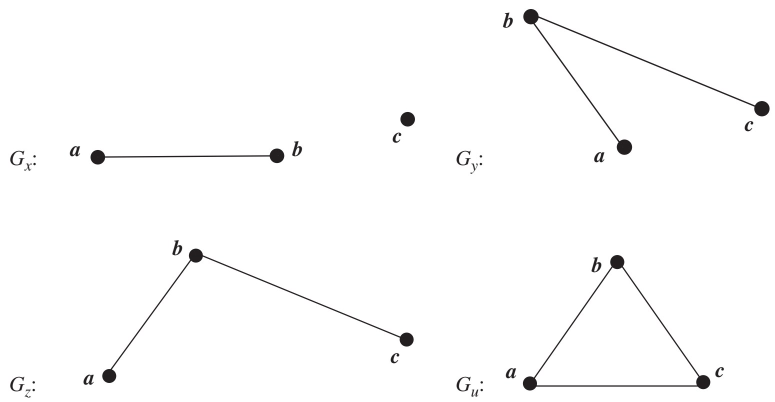
3. The primary decomposition

For a graph G let G_x be the induced graph on $V(G)\backslash\{x\}$. Equivalently, the graph G_x is formed by removing a vertex x (and its adjacent edges) from G. We call the collection of all such induced subgraphs the *deck* of G, denoted by $\mathscr{D}(G)$, and each G_x is called a *card* of the deck. If $I(G)$ is the edge ideal of G in the polynomial ring $R=k[x_1, \dots, x_n]$ and $S = R/I(G)$, then the edge ideal of G_x is the image of $I(G)$ under the quotient map $R \to R^x = R/(x)$. Let $S^x = R^x/(I(G)R^x)$.

Example 3.1. Let G be the graph given below:



The deck $\mathscr{D}(G)$ consists of the following cards:



Our goal, therefore, is to see what properties of the ideal $I(G) = (xy, xz, xu, yz)$ we can recover from the edge ideals of the cards: $I(G_x) = (ab)$, $I(G_y) = (ab, bc)$, $I(G_z) = (ab, bc)$, and $I(G_u) = (ab, bc, ac)$.

We will have occasion to use the following result from combinatorial reconstruction theory:

Lemma 3.2 (*Kelly's Lemma [6]*). *For simple graphs F and G let $S(F, G)$ denote the number of induced subgraphs of G that are isomorphic to F. If $\#V(F) < \#V(G)$, then $S(F, G)$ is a reconstructible invariant of G.*

Proof. Label the vertices of G, and fix a labeled subgraph isomorphic to F. Then this subgraph appears as a subgraph of $\#V(G) - \#V(F)$ cards. There are $S(F, G)$ labeled subgraphs isomorphic to F, so

$$S(F, G)(\#V(G) - \#V(F)) = \sum_{x \in V(G)} S(F, G_x).$$

Hence $S(F, G)$ is reconstructible. □

Lemma 3.3 (*Minimal vertex cover of graph and deck*). *Suppose that G is a graph with deck $\mathscr{D}(G)$, and let x be a vertex of G. Then*

(1) *If A is a minimal vertex cover of G containing x, $A' = A \backslash \{x\}$ is a minimal vertex cover for the card G_x.*
(2) *If A' is a minimal vertex cover of G_x, then either A' or $A = A' \cup \{x\}$ is a minimal vertex cover of G.*

Proof. (1) Every edge of G_x is an edge of G not incident to x. Since A is a vertex cover of G, these edges must be incident to a vertex in A, hence to a vertex in $A' = A\backslash\{x\}$. So A' is a vertex cover for G_x. Moreover, A' is minimal since if $B' \subsetneq A'$ is a minimal vertex cover of G_x, then $B' \cup \{x\} \subsetneq A$ is a subcover of A, contradicting the minimality of A.

(2) Suppose that A' is not a minimal vertex cover of G. Clearly A is a vertex cover of G. Suppose that there is a subset $B \subsetneq A$ covering G. Then $x \in B$, since otherwise, B would be a proper subset of A' covering G_x, which is a contradiction. So $B = B' \cup \{x\}$, with $B' \subsetneq A'$. By the previous part, B' is a minimal vertex cover of G_x, which is again a contradiction. So A has to be a minimal vertex cover of G. □

Remark 3.4. If A is a minimal vertex cover of G not containing a vertex x, A may or may not be a minimal vertex cover for the card G_x. For example, take the graph G with $I(G) = (xu, uy, yv)$. Then $\{y, x\}$ is a minimal vertex cover of both G and G_v, whereas $\{u, y\}$ is a minimal vertex cover of G, but not of G_v.

We summarize some algebraic facts about $I(G)$ in order to give an algebraic interpretation of the minimal vertex covers. Since the ideal $I(G)$ is radical, it is the intersection of the prime ideals containing it. In particular, $I(G)$ is the intersection of the primes containing $I(G)$ that are minimal with respect to inclusion. Moreover, since $I(G)$ is generated by monomials, each of its minimal primes is generated by a subset of the variables. The ideal $(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ has *height* equal to the number of variables it contains. The *Krull dimension* of $S = k[x_1, \ldots, x_n]/I(G)$ is

$$\dim S = n - (\text{minimum height of a minimal prime of } I(G)). \tag{1}$$

It follows from the definition of a minimal vertex cover that the sets of variables appearing in a minimal prime of $I(G)$ are precisely the minimal vertex covers of G.

The following is the algebraic translation of Lemma 3.3.

Corollary 3.5 (*Primary decomposition of a graph and its deck*). *Suppose that G is a graph and $I(G)$ its edge ideal.*

(1) *If $\mathfrak{p} = (x_1, \ldots, x_r)$ is a minimal prime ideal of height r containing $I(G)$, then the ideal $(x_1, \ldots, \hat{x}_i, \ldots, x_r)$ is a minimal prime ideal containing $I(G_{x_i})$. In this case, there are at least r cards in $\mathscr{D}(G)$ with a minimal prime of height $r-1$.*
(2) *If for some vertex x of G, $I(G_x)$ has a minimal prime $\mathfrak{p}$ of height $r-1$, then $I(G)$ has a minimal prime of height $r-1$ (namely $\mathfrak{p}$), or a minimal prime of height r (namely $\mathfrak{p} + (x)$).*

Suppose $H(G, r)$ denotes the number of height r minimal primes of the ideal $I(G)$. It follows directly from Corollary 3.5 that

Corollary 3.6 (*Number of components of fixed height*). *Let G be a graph. Then for each r*

$$H(G, r) \leqslant \frac{1}{r} \sum_{x \in G} H(G_x, r-1).$$

In particular, if d is the minimum possible height for a prime ideal of $I(G)$, then

$$H(G, d) = \frac{1}{d} \sum_{x \in G} H(G_x, d-1).$$

Proof. Part 1 just follows from part 1 of Corollary 3.5. For part 2, we know that a minimal prime $\mathfrak{p}$ of height $d-1$ of a card G_x could only come from a height d minimal prime $\mathfrak{p}+(x)$ of $I(G)$ (because there are no minimal primes of $I(G)$ of height $d-1$). Since each such minimal prime of $I(G)$ contributes to exactly d cards, our claim follows. $\square$

4. The Hilbert function

The Hilbert function of a graded ring $S=\oplus_{n\in\mathbb{N}}S_n$, where $S_0=k$ is a field, measures the growth of the graded components of S. To be precise, each graded piece S_n of S is a k-vector space and the *Hilbert series* of S is the generating function for the dimensions of these vector spaces:

$$H_S(m)=\sum_{n\in\mathbb{N}}\dim_k S_n m^n.$$

The *Hilbert function* of S is the function $H(S,m)=\dim_k S_n$.

As before, let G be a graph and $S=R/I(G)$. We will show that the Hilbert function of S is reconstructible.

It is well known [2, Theorem 5.1.7]) that the Hilbert series of S can be represented in terms of the f-vector $(f_0,\ldots,f_d)$ of the Stanley–Reisner complex Δ_G:

$$H_S(m)=\sum_{i=-1}^{d}\frac{f_i m^{i+1}}{(1-m)^{i+1}},$$

where $f_{-1}=1$ is the number of empty subgraphs in G. It follows that the Hilbert function of S can be described as

$$H(S,m)=\begin{cases}1 & \text{if } m=0,\\ \sum_{i=0}^{d} f_i\binom{m-1}{i} & \text{if } m>0.\end{cases} \tag{2}$$

A set A of vertices of a graph G is called *independent* if no two elements of A are connected by an edge in G. Let $f_s(G)$ be the number of independent sets of size $s+1$ in G, in other words $(f_0,f_1,\ldots,f_d)$ is the f-vector of the Stanley–Reisner complex Δ_G of $I(G)$.

Lemma 4.1. *For a graph G with more than two vertices, the f-vector of the Stanley–Reisner complex of $I(G)$ is reconstructible.*

Proof. Let $\overline{K_s}$ be the complement of the complete graph K_s, that is, $\overline{K_s}$ is the graph on s vertices with no edges. Notice that $f_s(G)$ is the number of instances of the graph $\overline{K_s}$ in G. If $s<n$ this number is reconstructible by Lemma 3.2.

If $s=n$ the number is zero or one depending on whether G is the empty graph of not, and since the empty graph on more than two vertices is reconstructible, so is $f_s(G)$. $\square$

Since the Hilbert function can be described in terms of the f-vector as demonstrated in (2), we conclude that the Hilbert function is reconstructible.

Proposition 4.2 (*Hilbert function is reconstructible*). *The Hilbert function of $I(G)$ for a given graph G with more than two vertices is reconstructible.*

As immediate corollaries of the above facts, we can show that the multiplicity and Krull dimension (see (1)) of $I(G)$ are also reconstructible.

Corollary 4.3 (*Dimension is reconstructible*). *For a graph G with more than two vertices, the (Krull) dimension of $S=R/I(G)$ is reconstructible.*

Proof. The Krull dimension of S is equal to $\dim\Delta_G+1$ ([2, Theorem 5.1.4]), which we can immediately compute from the f-vector of Δ_G or the Hilbert function of S. $\square$

The *multiplicity* of S is an invariant that can be described in terms of the coefficients of the numerator of the Hilbert function, and in the case we are dealing with, is the same as f_d, the number of d-dimensional facets of Δ_G [2, Chapter 5]). Hence, this value is also reconstructible, as the f-vector (or the Hilbert function) is reconstructible.

Corollary 4.4 (*Multiplicity is reconstructible*). *Given a graph G with more than two vertices, the multiplicity of $S = R/I(G)$ is reconstructible.*

Remark 4.5. The results of Section 3 provide a different proof for reconstruction of dimension and multiplicity. In particular one can deduce $\dim S = \max\{\dim S^x\}$.

Remark 4.6. Notice that Corollary 4.4 implies that we can reconstruct the number of components of *Spec*(S) of maximal dimension. This raises the question of whether the number of the components of a given non-maximal dimension is also reconstructible. In particular it would be of interest to know whether unmixedness is recognizable. That is, given the deck of a graph G, can we determine whether all the minimal primes of $I(G)$ have the same height?

5. Graded Betti numbers

Given a monomial ideal I in $R = k[x_1, \ldots, x_n]$, we define the multigraded Betti numbers $\beta_{i,\mathbf{b}}$ of I in terms of a multigraded minimal free resolution of $S = R/I$ as an R-module:

$$0 \leftarrow R/I \leftarrow R \leftarrow \oplus_{\mathbf{b}} R(-\mathbf{b})^{\beta_{1,\mathbf{b}}} \leftarrow \oplus_{\mathbf{b}} R(-\mathbf{b})^{\beta_{2,\mathbf{b}}} \leftarrow \ldots,$$

where the modules $R(-\mathbf{b})$ are shifts of the polynomial ring R to make the differentials in the resolution multidegree-preserving maps. As well, we define $\beta_{i,j} = \sum_{|\mathbf{b}|=j} \beta_{i,\mathbf{b}}$, where $|\mathbf{b}| = b_1 + \cdots + b_n$. When I is a squarefree monomial ideal (as is the case for $I(G)$) then each $\mathbf{b}$ appearing in the resolution is also squarefree in the sense that $\mathbf{b} \in \{0, 1\}^n$ (see [2, Section 5.5]).

We will use Hochster's formula from Stanley–Reisner theory to study the multigraded Betti numbers of the edge ideal of a graph. We will prove that for a graph G on n vertices the graded Betti numbers $\beta_{i,j}$ are reconstructible for all $j < n$. The reconstruction of the top degree Betti numbers $\beta_{i,n}$, on the other hand, seems to be very difficult.

Suppose that $I = I(G)$ is the edge ideal of a graph G, and let Δ denote the Stanley–Reisner complex of I. Suppose $\mathbf{b} \in \mathbb{Z}^n$ is a vector consisting of 0's and 1's, and let $B = \{x_i \mid \mathbf{b}_i = 1\}$ be the support of $\mathbf{b}$. Then by Hochster's formula (see [2, Chapter 5]), the multigraded Betti number $\beta_{i,\mathbf{b}}$ of R/I can be computed via the reduced simplicial homology of certain subcomplexes of Δ:

$$\beta_{i,\mathbf{b}} = \beta_{i,B} := \dim_k \widetilde{H}_{|B|-i-1}(\Delta_B; k), \tag{3}$$

where Δ_B denotes the restriction of Δ to B; in other words, $\Delta_B = \{F \in \Delta \mid F \subseteq B\}$. A simple unraveling of the definitions shows that Δ_B is just the Stanley–Reisner complex of $I(G_B)$.

Theorem 5.1 (*Reconstruction of graded Betti numbers*). *Let G be a graph on n vertices. Then the graded Betti numbers $\beta_{i,j}$ of $I(G)$ are reconstructible for all $j < n$.*

Proof. Let $\mathscr{D}$ be a deck with cards labeled $G_{x_1}, \ldots, G_{x_n}$, and let G be a reconstruction of $\mathscr{D}$. Let Δ denote the Stanley–Reisner complex of $I(G)$, and Δ^x the Stanley–Reisner complex of G_x, for a vertex x of G. Suppose that $B \subsetneq V = \{x_1, \ldots, x_n\}$. If $x \notin B$, then removing x and then restricting to B has the same effect as just restricting to B:

$$(\Delta^x)_B = \Delta_B.$$

Applying Hochster's formula (3), if $\beta^x_{i,B}$ denotes the multigraded Betti number of $R^x/I(G_x)$, and if $x \notin B$, then

$$\beta_{i,B} = \dim_k \widetilde{H}_{|B|-i-1}(\Delta_B; k) = \dim_k \widetilde{H}_{|B|-i-1}(\Delta^x_B; k) = \beta^x_{i,B}.$$

Since $\beta_{i,j} = \sum_{B \subseteq V, |B|=j} \beta_{i,B}$, we are done. $\square$

Remark 5.2 (*The top degree Betti number*). Calculating the degree n graded Betti number $\beta_{i,n}$, where n is the number of variables, seems to be much more difficult. From the fact that the Hilbert series is reconstructible, it follows that the alternating sum $\sum_i (-1)^i \beta_{i,n}$ is reconstructible. In light of this, it would be helpful if we knew that the top degree Betti numbers occurred only in one spot of the resolution. However, this is not always true. The example below illustrates this fact; similar examples can be constructed with the top Betti numbers in different columns of the Betti diagram.

In the case where we know that $I(G)$ is a Cohen–Macaulay ideal, though, the top degree Betti number can appear only at the last slot of the resolution [10, Proposition 4.2.3]. Therefore in this case all graded Betti numbers are reconstructible. We would like to thank Hossein Sabzrou for pointing this out to us.

Example 5.3. Let $R = k[x_1, \ldots, x_9]$, and

$$I = (x_1x_2, x_3x_4, x_5x_6, x_7x_8, x_9x_1, x_9x_2, x_9x_3, x_9x_4, x_9x_5, x_9x_6, x_9x_7, x_9x_8).$$

Macaulay 2 [3] returns the following Betti diagram for R/I.

total:	1	12	38	66	75	57	28	8	1
0:	1	.	.	.	.	.	.	.	.
1:	.	12	32	56	70	56	28	8	1
2:	.	.	6	6	.	.	.	.	.
3:	.	.	.	4	4	.	.	.	.
4:	.	.	.	.	1	1	.	.	.

To read this diagram, assume that the rows and columns are numbered 0, 1, 2, ... Then the entry in the ith row and jth column is the Betti number $\beta_{j,i+j}$.

We see in the diagram that $\beta_{5,9} = \beta_{8,9} = 1$, so the degree 9 Betti number happens in two different spots of the resolution.

Note that the highest degree graded Betti number controls many other invariants of $I(G)$.

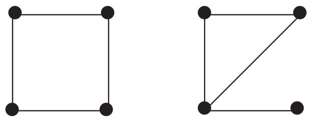
Proposition 5.4. *Let G be a graph on n vertices. If the graded top-degree Betti numbers $\beta_{i,n}$ of $I(G)$ are reconstructible, then the algebraic invariants depth, projective dimension and regularity of $I(G)$ are reconstructible.*

Proof. We already know from Theorem 5.1 that all other graded Betti numbers are reconstructible. The projective dimension of $I(G)$ is the maximum i such that $\beta_{i,j} \neq 0$, and so once we know all the Betti numbers, we know the projective dimension. One can then apply the Auslander–Buchsbaum formula (see [2, Chapter 1]) to compute the depth from the projective dimension. The regularity of $I(G)$ is defined as the maximum value of $j - i$ where the graded Betti number $\beta_{i,j} \neq 0$. So once again, reconstructing all the Betti numbers will lead to reconstructing the regularity of $I(G)$. $\square$

A natural question is: Can these invariants be reconstructed independent of the reconstruction of the top-degree Betti numbers?

The following example shows that we cannot hope to have uniqueness of ideals associated to Betti diagrams, even if the ideals are edge ideals.

Example 5.5. Consider the following two graphs.



Their edge ideals have the same Betti diagram, even though the graphs are different.

total:	1	4	4	1
0:	1	.	.	.
1:	.	4	4	1

In [5] Katzman constructed a graph G for which the Betti diagram of G depends on the characteristic of the ground field. If H is a smallest such graph, then the Betti diagrams of $H\setminus\{v\}$ are all characteristic independent, while the Betti diagram of H itself depends on the characteristic of the ground field. Therefore the knowledge of the Betti diagrams of the cards in the deck of a graph is not sufficient to reconstruct the Betti numbers of the graph. However if the ground field is known this becomes an interesting question:

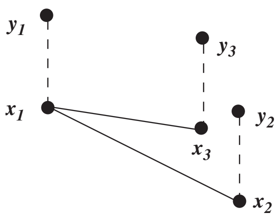
Question 5.6. Suppose that the field k and the Betti diagrams of all the cards in the deck of G are given. Can the Betti diagram of G be reconstructed from this information?

6. Suspended graphs

Suspended graphs (see [9,10]) provide an important set of examples of Cohen–Macaulay graphs. It is known that a tree is Cohen–Macaulay if and only if it is suspended (if and only if it is unmixed; see [9]).

Definition 6.1 (*Suspension*). Let G be a graph on n vertices $\{x_1, \dots, x_n\}$. The *suspension* of G is the graph obtained by attaching to each vertex x_i the new vertex y_i and the edge $\{x_i, y_i\}$. We call a graph G *suspended* if it is the suspension of another graph.

Example 6.2. The graph below is the suspension of the graph G with $I(G) = (x_1x_2, x_1x_3)$.



Motivated by the goal of investigating the reconstructibility of the Cohen–Macaulay graphs we prove that suspended graphs are recognizable and reconstructible.

Lemma 6.3. *Suppose that G is a connected graph with $2n > 2$ vertices. The number of cards with exactly one isolated vertex in the deck of G is equal to n if and only if G is suspended.*

Proof. If G is suspended then the cards in the deck corresponding to removal of degree one vertices are connected. And the cards in the deck corresponding the removal of any of the other vertices have exactly one isolated vertex, i.e. the degree one vertex suspending the removed vertex.

Conversely suppose there are n cards in the deck of a connected graph G, each with exactly one isolated vertex. These cards correspond to removal of n vertices $x_1, x_2, \dots, x_n$. We claim each one of the x_i's has exactly one neighbor with degree one, and none of the x_i's have degree one. The first part of the claim follows from the fact that removing x_i produces a card with exactly one isolated vertex, and the second part from the fact that G is connected. So the $2n$ vertices of G are partitioned into two sets, the x_i's and their degree one neighbors. So G is suspended. $\square$

As a corollary we get that suspended graphs are recognizable.

Corollary 6.4. *Suspended graphs are recognizable.*

Proof. Our goal is to prove that given the deck of a graph G, it is possible to decide whether G is suspended or not. Since disconnected graphs are recognizable and reconstructible, if the deck is the deck of a disconnected graph then by reconstructing the graph we can decide whether it is suspended or not. If the deck is the deck of a connected graph then using the previous lemma we can decide whether or not it is suspended. $\square$

Theorem 6.5 (*Reconstruction of suspended graphs*). *Suspended graphs with more than two vertices are reconstructible.*

Proof. Since we have proved that suspended graphs are recognizable, and since disconnected graphs are reconstructible we may assume we have a deck known to belong to a connected suspended graph G and attempt to reconstruct G. To do this we first reconstruct the degree sequence of G. We consider two separate cases:

Case 1. The highest degree in the degree sequence of G is 2. Suppose G is a suspension of a graph G'. By our assumption, every vertex of G' is of degree 1, and since G' is connected, it can only be the graph consisting of one edge. So in this case G is just a path of length 3.

Case 2. The highest degree is $d > 2$. Let $d_1 = d, d_2, \ldots, d_{2n-1}, d_{2n} = 1$ be the degree sequence of G. Also let v be a vertex of degree d and let w be its degree one neighbor. Removing w will produce a card whose degree sequence is a rearrangement of the sequence $d_1 - 1, d_2, d_3, \ldots, d_{2n-1}$ in non-increasing order. Pick a card with this degree sequence. Among all vertices of degree $d - 1$ in this card exactly one does not have a degree one neighbor, that vertex is v and adding a new vertex of degree one connected to v will result in a graph isomorphic to G. $\square$

If G is a suspended graph then $R/I(G)$ is Cohen–Macaulay. So a class of Cohen–Macaulay graphs is reconstructible. A natural question is: Are all Cohen–Macaulay graphs reconstructible?

Another natural and difficult question is: Which reconstructible algebraic parameters would imply the reconstruction conjecture?

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