

Differential Operators and Nakai's Conjecture

by

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Abstract

This thesis is concerned with differential operators on algebraic varieties. The main goal is to understand how the presence of singularities is reflected in the algebraic structure of the ring of differential operators. Motivated by applications to tight closure, we study varieties in arbitrary characteristic. The thesis focuses on the study of ideals stable under differential operators and applies results in this area to Nakai's conjecture and the theory of tight closure. As well, results on differential operators on étale extensions are applied to show that the ring of differential operators on a smooth variety is equal to the Hasse-Schmidt algebra. Along the way, the Hasse-Schmidt derivations are shown to extend over étale extensions.

The ring of differential operators on a Stanley-Reisner ring R of arbitrary characteristic is explicitly described in terms of the minimal primes of R . As well, the D-module structure of R is completely determined.

The structure of a k -algebra R as a module over the ring of differential operators $D(R/k)$, the Hasse-Schmidt algebra $HS(R/k)$ and the derivation algebra $der(R/k)$ is investigated. In particular, the conductor of the normalization R' of R into R is $HS(R/k)$ -stable and Hasse-Schmidt derivations on R extend to R' . This is the key observation in the proof that a characteristic-free analogue of Nakai's conjecture holds for varieties whose normalization is smooth.

Differential operators can also be applied to Hochster and Huneke's theory of tight closure. We show that Stanley-Reisner rings are Frobenius-split by constructing an *explicit* splitting of Frobenius in terms of differential operators. The D-module structure of a Stanley-Reisner ring R is used to easily compute the test ideal of R , recovering a result originally due to Cowden.

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Chapter 1

Introduction

1.1 Context and summary of main results

This thesis is concerned with the study of rings of differential operators on algebraic varieties in the sense described by Grothendieck [12, chapter 16]. Rings of differential operators are one of the most important noncommutative (associative) algebras. They play an important role in the representation theory of Lie algebras [2] and the algebraic analysis of systems of partial differential equations as developed by Bernstein [3] and Kashiwara [26]. The aim of this thesis is to use the techniques of commutative algebra to study the D-module structure of the coordinate ring of a singular affine variety and to study the algebraic structure of its ring of differential operators. The objective is to better understand how the nature of singularities of a variety is reflected in the structure of its ring of differential operators.

1.1.1 Background on operators

Background material on differential operators is included in chapter 2. This material is well-known in characteristic zero, but there is no complete treatment of the prime characteristic case. Following Grothendieck [12], the ring of differential operators $D(R/k)$ and the modules of higher differentials are defined relative to an arbitrary commutative k -algebra R . In particular, we do not assume that R is smooth over k and if k is a field, then it may be of arbitrary characteristic. The differential operators are continuous in every I -adic topology on R and this property characterizes differential operators on an algebra of finite type over an algebraically closed field.

Chapter 2 also contains background information on the Hasse-Schmidt derivations (also called higher order derivations in the literature), studied by Ribenboim [39, 40], Brown [6] and Heerema [13]. The Hasse-Schmidt algebra $HS(R/k)$ is introduced as an analogue of the algebra $der_k(R)$ generated by k -derivations; $HS(R/k)$ agrees with $der_k(R)$ in characteristic zero and is better behaved in prime characteristic. The exponential map sending a Hasse-Schmidt derivation to a ring endomorphism of $R[[t]]$ is used to characterize Hasse-Schmidt derivations (Theorem 2.2.6).

The Weyl algebra is introduced in chapter 2 as the ring of differential operators on a polynomial ring over a field. With this definition, the Weyl algebra is simple in any characteristic and equals its Hasse-Schmidt subalgebra. As well, the polynomial ring is D-simple; that is, it is a simple module over its ring of differential operators. The following two theorems from chapter 3 generalize this result.

Theorem 3.3.1. *Let R be a smooth algebra of finite type over a field k . Then the ring of differential operators equals the Hasse-Schmidt algebra:*

$$D(R/k) = HS(R/k).$$

Theorem 3.3.4. *If R is a smooth algebra over a field k , then R is a finite product of D -simple domains.*

These generalize a result due to Grothendieck [12, 16.11.2] and appear to be unpublished folklore. The proof of these results involves a detailed study of differential operators on étale extensions. Our presentation of this investigation is based on Másson's doctoral thesis [30]. Our contribution includes a study of the behavior of Hasse-Schmidt derivations under étale extensions. The following theorem extends results of Brown [5] and Ribenboim [40] on the behavior of Hasse-Schmidt derivations under localization.

Theorem 3.2.6. *Let $A \xrightarrow{f} B$ be a formally étale extension of k -algebras and suppose that $\Delta = \{\delta_n\} \subset HS(A/k)$ is a Hasse-Schmidt derivation on A . Then there exists a unique Hasse-Schmidt derivation $\gamma = \{\gamma_n\}$ on B such that $f \circ \delta_n = \gamma_n \circ f$.*

We also use the characterization of Hasse-Schmidt derivations on R in terms of ring maps on $R[[t]]$ to answer a question of Brown and Kuan [7]: not every Hasse-Schmidt derivation on a localization of R arises as the extension of a Hasse-Schmidt derivation on R .

1.1.2 D-module structure of R

The D-module structure of R is a powerful tool in understanding the algebraic structure of both the ring R and its ring of differential operators $D(R)$. In chapter 4, we treat the structure of R as a module over several natural rings of operators: $der(R)$, $HS(R)$ and $D(R)$. The exponential map intro-

duced in chapter 2 is used to show that Hasse-Schmidt stable ideals behave particularly well with respect to primary decomposition.

Theorem 4.2.4. *The associated primes of $HS(R)$ -stable ideals are $HS(R)$ -stable and every $HS(R)$ -stable ideal admits a primary decomposition with $HS(R)$ -stable components.*

Hasse-Schmidt derivations also behave well with respect to normalization.

Theorem 4.3.2. *Let R be a reduced Noetherian k -algebra and let R' be its integral closure in the total quotient ring L of R . Every Hasse-Schmidt derivation on R extends to a Hasse-Schmidt derivation on R' .*

Corollary 4.3.3. *The conductor of a reduced Noetherian k -algebra R is $HS(R/k)$ -stable.*

These last three results extend important results of Seidenberg [43, 44] on derivations. A detailed study of the D-module structure of R has implications for our treatment of Nakai's conjecture. Corollary 4.3.3 lays the foundation for our assault on Nakai's conjecture in chapter 6.

The D-module structure of R ought to have some bearing on the algebraic structure of R . Brown [6] showed that when the ring of differential operators on a \mathbb{C} -algebra R is generated by derivations, certain D-module conditions imply that R is normal. In chapter 6, our results on Nakai's conjecture are used to extend Brown's result to arbitrary characteristic.

Theorem 6.3.2. *Let R be a finitely generated reduced algebra over a field k and suppose that:*

- (1) $D(R/k) = HS(R/k)$ and
- (2) every D-stable prime ideal of R has height less than or equal to 1.

Then R is normal.

1.1.3 Computing differential operators

The ring of differential operators $D(R/\mathbb{C})$ on a smooth complex affine variety $X = \text{Spec}(R)$ agrees with the R -algebra generated by the derivations on R (all varieties are assumed to be reduced but not necessarily irreducible: R is a reduced \mathbb{C} -algebra of finite type). As such, the algebraic structure of these rings is well understood (see, for example, McConnell and Robson [33]). For example, $D(R/\mathbb{C})$ is a finitely generated, simple Noetherian domain.

In contrast to the smooth case, it may be difficult to describe the algebraic structure of the ring of differential operators on a singular variety. For example, Bernstein, Gelfand and Gelfand [4] showed that the ring of differential operators on the affine cone over a particular plane cubic curve is not finitely generated. In general, it is very difficult to compute rings of differential operators. In chapter 5, the ring of differential operators on a Stanley-Reisner ring R defined over an arbitrary commutative domain k is explicitly described. Such a ring R comes equipped with a monomial grading; this induces a powerful grading on $D(R/k)$. In turn, this facilitates the description of $D(R/k)$ in terms of the minimal primes of R .

Theorem 5.2.3. *Let R be a Stanley-Reisner ring defined over a commutative domain k . Then $D(R/k)$ is generated as an R -module by*

$$\{x^{\mathbf{b}}\partial^{\mathbf{a}} : x^{\mathbf{b}} \in P \text{ or } x^{\mathbf{a}} \notin P \text{ for each minimal prime } P \text{ of } R\},$$

and the nonzero operators of this form determine a free basis for $D(R/k)$ as a k -module. The condition on $x^{\mathbf{b}}\partial^{\mathbf{a}}$ above is equivalent to

$$x^{\mathbf{b}} \in \bigcap_{x^{\mathbf{a}} \in P_j} P_j,$$

where the P_j are the minimal primes of R .

This extends Brumatti and Simis’s description [8] of the derivations on a Stanley-Reisner ring of arbitrary characteristic. This theorem was independently established by Tripp [55] in the case that k is a field of characteristic zero using different methods. The ring of differential operators on R can also be described in terms of the local cohomology of R (Theorem 5.3.6) or the simplicial complex associated to R (Theorem 5.3.4). We also completely describe the D -module structure of a Stanley-Reisner ring defined over a field in terms of its minimal primes.

Theorem 5.4.5. *The D -submodules of a Stanley-Reisner ring R defined over a field k are precisely the intersections of sums of minimal primes of R .*

1.1.4 Nakai’s conjecture

Stafford and Smith [52] connect the local geometry of a complex curve $X = \text{Spec}(R)$ with the structure of its ring of differential operators: the normalization map $\tilde{X} \rightarrow X$ is injective if and only if $D(R/\mathbb{C})$ is a simple ring. Similarly, Nakai’s conjecture relates the algebraic structure of the ring of differential operators on a finitely generated \mathbb{C} -algebra R to the presence of singularities in $\text{Spec}(R)$. Inspired by Grothendieck’s observation that the ring of differential operators on a smooth complex variety is generated by derivations, Nakai conjectured the converse: the ring of differential operators on a complex variety is generated by derivations if and only if the variety is nonsingular. Nakai appears not to have stated the conjecture in the literature but it is often quoted in connection with his paper [38]. The first statement of the conjecture appears in [37]. Ishibashi [23] restated the conjecture in a characteristic-free manner by using the Hasse-Schmidt

derivations: an affine variety $\text{Spec}(R)$ defined over an algebraically closed field k is nonsingular if and only if the ring of differential operators $D(R/k)$ equals the Hasse-Schmidt algebra, $HS(R/k)$.

As we have already remarked, Grothendieck's result can be extended to the situation where k is an arbitrary field (Theorem 3.3.1). As well, since the ring of differential operators on R over k is a relative object, it makes more sense to relate the structure of $D(R/k)$ to the smoothness of the extension $k \rightarrow R$. This leads to the following more general statement of the conjecture.

Nakai's Conjecture: Let $X = \text{Spec}(R)$ be a variety defined over a field k ; then X is smooth over k if and only if $D(R/k) = HS(R/k)$.

Nakai's conjecture is known to imply the Zariski-Lipman conjecture: if $X = \text{Spec}(R)$ is a complex variety and the module of derivations, $Der_{\mathbb{C}}(R)$, is a locally free R -module, then R is nonsingular. We record a proof of this result (originally due to Becker and Rego [1]) in chapter 6.

Nakai's conjecture remains open though it is known to hold for several important classes of varieties. By relating the structure of the ring of differential operators on a curve to the ring of differential operators on its normalization, Mount and Villamayor [37] established the conjecture for complex curves. Ishibashi [25] established the conjecture for characteristic zero rings of invariants of finite groups. Schwarz [42] later proved substantially more general results. As well, Ishibashi [24] proved that Nakai's conjecture holds for two-dimensional weighted complete intersections in characteristic zero.

We establish a very general case of Nakai's conjecture that does not require any assumptions on the dimension of R .

Theorem 6.2.4. *Let R be a reduced algebra over a field k . If $HS(R/k) =$*

$D(R/k)$ and the normalization R' of R is smooth over k , then R is smooth over k .

In particular, Nakai's conjecture holds for curves of arbitrary characteristic, extending the result of Mount and Villamayor [37].

Corollary 6.2.5. *Let R be a reduced algebra over a field k . If R is 1-dimensional and $HS(R/k) = D(R/k)$ then R is smooth over k .*

Theorem 6.2.4 can also be used to give a simple proof that Nakai's conjecture holds for Stanley-Reisner rings, extending a result of Schreiner [41]. More generally, the theorem implies that Nakai's conjecture holds for varieties all of whose components are smooth.

Corollary 6.2.6. *Let R be a reduced algebra over a field k . If $HS(R/k) = D(R/k)$ and if $\frac{R}{P}$ is smooth over k for each minimal prime P of R , then R is smooth over k . In particular, if R is a Stanley-Reisner ring and $HS(R/k) = D(R/k)$ then R is a polynomial ring.*

1.1.5 Tight closure

It turns out that the differential operators on a variety $X = \text{Spec}(R)$ of finite type over a perfect field of characteristic p are just the endomorphisms of R that are R^{p^e} -linear for sufficiently high Frobenius powers of R . This suggests a connection to tight closure, a technique in commutative algebra that uses the Frobenius morphism to assign to each ideal I in a ring of prime characteristic a possibly larger ideal, its tight closure I^* . Tight closure, together with the technique of reduction to characteristic p , can be used to prove characteristic zero results whose statement does not involve tight closure at all. For instance, these methods give a very elegant proof of the

theorem of Hochster and Roberts [19] that rings of invariants of linearly reductive groups are Cohen-Macaulay (see [16]). As well, tight closure has applications to homological algebra and singularity theory.

Smith [48, 47] was the first to articulate the philosophy that differential operators can be used to study tight closure. Chapter 5 ends with a survey of Smith's characterization of strongly F-regular rings in terms of their D-module structure. Chapter 7 illustrates another instance of this approach: differential operators are used to study tight closure on Stanley-Reisner rings. We begin by showing that differential operators give an easy constructive proof of Hochster and Robert's observation [20] that Stanley-Reisner rings defined over a field of prime characteristic are F-split.

Theorem 7.1.1. *If $R = \frac{k[x_1, \dots, x_n]}{I}$ is a Stanley-Reisner ring of prime characteristic, then the inclusion map $R^p \hookrightarrow R$ is split by the differential operator $\frac{1}{(p-1)!^n} \frac{\partial^{p-1}}{\partial x_1^{p-1}} \cdots \frac{\partial^{p-1}}{\partial x_n^{p-1}} x_1^{p-1} \cdots x_n^{p-1}$.*

We also produce an easy proof that tight closure commutes with localization in Stanley-Reisner rings. It is worthwhile to point this out in light of the difficulty encountered in proving this in general. For different perspectives on this result, see Smith and Swanson [50] and Katzman [27]. This result has been extended to a more general class of rings (including the coordinate ring of any variety defined by binomial equations) by Smith [46].

Corollary 7.1.4. *Tight closure commutes with localization in a Stanley-Reisner ring.*

Part of the motivation for studying the D-module structure of a Stanley-Reisner ring R was Smith's observation that the test ideal of R is D-stable. The description of the D-module structure of R in Theorem 5.4.5 can be used to compute the test ideal of a Stanley-Reisner ring. This gives a new

proof of a result due to Cowden [10].

Theorem 7.1.7. *For a Stanley-Reisner ring R with minimal primes P_1, \dots, P_r , the test ideal of R is $\sum_{i=1}^r P_1 \cap \dots \cap \hat{P}_i \cap \dots \cap P_r$.*

Chapter 2

Rings of Operators

2.1 The ring of differential operators

This chapter summarizes the elementary theory of rings of differential operators. Much of this material is treated in various places in the research literature. The reader's attention is drawn to the work of Grothendieck [12], Miličić [35] and McConnell and Robson [33]. Unfortunately, these only treat the characteristic zero theory and only Grothendieck [12] deals with singular varieties.

The Hasse-Schmidt derivations are also introduced. These were extensively studied by Ribenboim [39, 40], Brown [6] and Heerema [13].

2.1.1 Differential operators

Let X be an affine algebraic variety defined over a commutative domain k . Let R be the coordinate ring of X : $X = \text{Spec}(R)$. The ring $R = H^0(X, \mathcal{O}_X)$ can be identified with the commutative ring of regular functions from X to k . Following Grothendieck [12], we can also associate to the k -variety X a non-

commutative ring, its ring of differential operators. Equivalently, the ring of differential operators $D(R/k)$ is associated to the extension $k \rightarrow R$. When no confusion might arise, we sometimes drop k from the notation and write $D(R)$ for the ring of differential operators. The ring of differential operators consists of certain k -linear endomorphisms of R : $D(R/k) \subseteq \text{End}_k(R)$.

The subring $D(R/k)$ of $\text{End}_k(R)$ can be identified in terms of the natural action of $R \otimes_k R$ on $\text{End}_k(R)$: $a \otimes b$ acts on a map $\theta : R \rightarrow R$ to give the map

$$a \circ \theta \circ b : R \xrightarrow{b} R \xrightarrow{\theta} R \xrightarrow{a} R.$$

Here we have identified the elements a and b of R with the endomorphisms of R given by multiplication by a and b , respectively. In what follows, we continue to make this identification without further comment. In order to distinguish $d \circ a$ and the image of a under a differential operator d , we write the former as da and the latter as $d * a$ or $d(a)$. The ring $R \otimes_k R$ is the coordinate ring of the product variety $X \times_{\text{Spec}(k)} X$. Let $J \subset R \otimes_k R$ be the ideal defining the diagonal $\Delta \subset X \times X$.

Lemma 2.1.1. *The ideal J is the kernel of the multiplication map*

$$\begin{aligned} R \otimes_k R &\rightarrow R \\ a \otimes b &\mapsto ab, \end{aligned}$$

and J is generated by all elements of the form $1 \otimes a - a \otimes 1$, where $a \in R$.

Proof. The elements $1 \otimes a - a \otimes 1$ are clearly in the kernel of the multiplication map. Suppose that $\sum a_i b_i = 0$ (that is, $\sum a_i \otimes b_i$ is in the kernel of the multiplication map), then

$$\begin{aligned} \sum a_i \otimes b_i &= (\sum a_i \otimes b_i) - (\sum a_i b_i \otimes 1) \\ &= \sum (a_i \otimes b_i - a_i b_i \otimes 1) \\ &= \sum (a_i \otimes 1)(1 \otimes b_i - b_i \otimes 1). \end{aligned}$$

So $J = (1 \otimes a - a \otimes 1)_{a \in R}$. Now it is clear that J defines the subscheme Δ . □

Note that a generator $(1 \otimes a - a \otimes 1)$ of J acts on an endomorphism $\theta \in \text{End}_k(R)$ to produce a commutator:

$$(1 \otimes a - a \otimes 1)\theta = \theta a - a\theta = [\theta, a].$$

The multiplication maps determined by $r \in R$ are precisely those endomorphisms of R killed by J . To see this, note that $[r, a] = 0$ for all $a \in R$ because R is a commutative ring. Also, if $[\theta, a] = 0$ for all $a \in R$, then θ is R -linear and so θ is just multiplication by $\theta * 1$.

The k -derivations of R are also related to the $R \otimes_k R$ action on $\text{End}_k(R)$. Suppose that d is a k -derivation of R , $d \in \text{Der}_k(R)$. For a and b in R ,

$$\begin{aligned} d * (ab) &= (d * a)b + a(d * b) \\ \iff d * (ab) - a(d * b) &= (d * a)b \\ \iff (da - ad) * b &= (d * a)b \\ \iff [d, a] * b &= (d * a)b \\ \iff [d, a] &= (d * a) \\ \implies J([d, a]) &= 0 \\ \implies J^2 d &= 0. \end{aligned}$$

So derivations are killed by J^2 . Of course, multiplication maps are also killed by J^2 . In fact, the endomorphisms of R killed by J^2 form an R -submodule of $\text{End}_k(R)$ isomorphic to $\text{Der}_k(R) \oplus R$. If $J^2\theta = 0$, then $\theta - \theta * 1$ is a

derivation:

$$\begin{aligned}
(\theta - \theta * 1)(ab) &= \theta(ab) - (\theta * 1)(ab) \\
&= [\theta, a](b) + a\theta(b) - (\theta * 1)(ab) \\
&= ([\theta, a] * 1)(b) + a\theta(b) - (\theta * 1)(ab) \\
&= \theta(a)b - a(\theta * 1)b + a\theta(b) - (\theta * 1)(ab) \\
&= \theta(a)b + a\theta(b) - 2(\theta * 1)(ab) \\
&= (\theta - \theta * 1)(a)b + a(\theta - \theta * 1)(b).
\end{aligned}$$

Now $\theta = (\theta - \theta * 1) + (\theta * 1) \in \text{Der}_k(R) \oplus R$.

Extending this behavior, we say that an endomorphism $\theta \in \text{End}_k(R)$ is a differential operator of order $\leq n$ if and only if $J^{n+1}\theta = 0$. We write $D^n(R/k)$ for the R -module of differential operators of order $\leq n$. The ring of differential operators $D(R/k)$ is just the direct limit (union) of these submodules of $\text{End}_k(R)$:

$$D(R/k) = \{\theta \in \text{End}_k(R) : J^m\theta = 0 \text{ for some } m\}.$$

Sometimes it is convenient to speak of differential operators of negative order. We set $D^n(R/k) = 0$ for $m < 0$.

A similar construction produces the R -module of differential operators $D(M, N) = D_{R/k}(M, N)$ between two modules M and N over a commutative k -algebra R . This is an R -submodule of $\text{Hom}_k(M, N)$. The ring $R \otimes_k R$ acts on $\text{Hom}_k(M, N)$: $a \otimes b$ acts on an R -module map $\theta : M \rightarrow N$ to give the map

$$a \circ \theta \circ b : M \xrightarrow{b} M \xrightarrow{\theta} N \xrightarrow{a} N.$$

A homomorphism $\theta \in \text{Hom}_k(M, N)$ is a differential operator of order $\leq n$ if $J^{n+1}\theta = 0$ and

$$D(M, N) = \{\theta \in \text{Hom}_k(M, N) : J^m\theta = 0 \text{ for some } m\}.$$

Many results about differential operators are proven by induction on order. In light of this, it is worth reiterating that θ is a differential operator of order $\leq n$ if and only if the commutator $[\theta, a]$ is a differential operator of order $\leq n - 1$ for all $a \in R$. This explains why the natural idea in many proofs is to consider commutators of differential operators with elements of R .

As yet we have only established that $D(R/k)$ is an R -module, but in fact it is a ring under composition of operators.

Theorem 2.1.2. *If $\delta \in D^n(R/k)$ and $\partial \in D^m(R/k)$ then $\delta \circ \partial \in D^{m+n}(R/k)$.*

Proof. By definition, if $\delta \in D^n(R/k)$ then

$$[\delta, a] = (1 \otimes a - a \otimes 1)\delta \in JD^n(R/k)$$

is a differential operator of order $\leq n - 1$. We compute:

$$\begin{aligned} (1 \otimes a - a \otimes 1)(\delta \circ \partial) &= \delta \circ \partial a - a\delta \circ \partial \\ &= \delta \partial a - \delta a \partial + \delta a \partial - a\delta \partial \\ &= \delta \circ [\partial, a] + [\delta, a] \circ \partial. \end{aligned}$$

Now the result follows by induction on $n + m$. □

This description of the ring of differential operators can be stated in the language of local cohomology. If J is an ideal of a commutative ring R and M is an R -module, then the *zeroth local cohomology of M with support in J* , $H_J^0(M)$, is the module of elements of M killed by some power of J :

$$H_J^0(M) = \{m \in M : J^n m = 0 \text{ for some } n\}.$$

It follows that the ring of differential operators $D(R/k)$ is just the zeroth local cohomology of the $R \otimes_k R$ -module $End_k(R)$ with support in the kernel

J of the multiplication map $R \otimes_k R \rightarrow R$:

$$D(R/k) = H_J^0(\text{End}_k(R)).$$

More generally, if M and N are R -modules,

$$D(M, N) = H_J^0(\text{Hom}_k(M, N)).$$

2.1.2 Modules of differentials

The ring R acts on $R \otimes_k R$ in each factor. When we speak of $R \otimes_k R$ as an R -module, we will always refer to the action of R on the first factor:

$$r(a \otimes b) = ra \otimes b.$$

The module $P_{R/k}^n = \frac{R \otimes_k R}{J^{n+1}}$ with R -module structure coming from the left factor of R is called the module of higher differentials of order $\leq n$ (sometimes $P_{R/k}^n$ is also called the module of n -jets). The module $P_{R/k}^n$ is not only an R -module, but also an R -algebra.

The map $d : R \rightarrow R \otimes_k R$ given by $d(a) = 1 \otimes a$ also plays an important role in the theory. In particular, the map $d_{R/k}^n : R \rightarrow P_{R/k}^n$ induced by d is a differential operator of R -modules of order $\leq n$. The map $d_{R/k}^n$ is called the universal derivation of order n . When no confusion will arise, we will just write d for $d_{R/k}^n$. The map $d : R \rightarrow P_{R/k}^n$ is a ring map, $d(rs) = d(r)d(s)$, but the reader is cautioned that d is not an R -algebra map.

We have the natural isomorphism:

$$D^n(M, N) = \text{Hom}_{R \otimes_k R}(P_{R/k}^n, \text{Hom}_k(M, N)) \cong \text{Hom}_R(P_{R/k}^n \otimes_R M, N),$$

where $P_{R/k}^n = \frac{R \otimes_k R}{J^{n+1}}$. Here, an R -module homomorphism $\phi : P_{R/k}^n \otimes M \rightarrow N$ is identified with the differential operator of order $\leq n$ given by

$$\begin{aligned} M &\xrightarrow{\cong} R \otimes M \xrightarrow{d \otimes 1} P_{R/k}^n \otimes M \xrightarrow{\phi} N. \\ m &\longmapsto 1 \otimes m \end{aligned}$$

Conversely, given any differential operator $\delta \in D^n(M, N)$, there is a unique map of R -modules $\phi_\delta : P_{R/k}^n \otimes M \rightarrow N$ such that $\delta = \phi_\delta \circ d_{R/k}^n$. It follows that $P_{R/k}^n \otimes M$, together with the map $R \xrightarrow{d_{R/k}^n \otimes 1} P_{R/k}^n \otimes M$, is characterized by a universal mapping property. We summarize this discussion in the following Proposition.

Proposition 2.1.3. *The R -module $P_{R/k}^n$ and the map $d_{R/k}^n : R \rightarrow P_{R/k}^n$ are the unique (up to isomorphism) pair (T, ∂) of R -module T and R -module map $\partial : R \rightarrow T$ such that given any differential operator $\delta \in D^n(M, N)$ between two R -modules M and N , there is a unique R -module map $\phi : T \otimes_R M \rightarrow N$ making the following diagram commute:*

$$\begin{array}{ccc}
 & & T \otimes M \\
 & \nearrow \partial \otimes 1 & \downarrow \phi \\
 R \otimes_R M \cong M & \xrightarrow{\delta} & N
 \end{array}$$

Lemma 2.1.4. *Let $R = k[x_1, \dots, x_n] = k[x]$ be a polynomial ring over the ring k . Then $P_{R/k}^t$ is the free R -module generated by the monomials of degree $\leq t$ in dx_1, \dots, dx_n .*

Because $P_{R/k}^n = \frac{R \otimes R}{J^{n+1}} = \frac{k[x_1, \dots, x_n, dx_1, \dots, dx_n]}{(x_1 - dx_1, \dots, x_n - dx_n)^{t+1}}$, where x_i corresponds to $x_i \otimes 1$ and dx_i corresponds to $1 \otimes x_i$, the proof is easy and is left to the reader.

2.1.3 Differential operators as continuous maps

Given an ideal I in a ring R , we consider the I -adic topology on R . A basis of open sets for this topology is given by the sets

$$a + I^n,$$

where $a \in R$ and $n \geq 0$. Differential operators on R are continuous in the I -adic topology:

Theorem 2.1.5. *Let I be an ideal in the ring R . If $\delta \in D^n(R)$, then $\delta * I^m \subseteq I^{m-n}$.*

Proof. This follows by induction on n . Since $D^0(R) = R$, the statement holds for $n = 0$. Now suppose that

$$D^n(R) * I^m \subseteq I^{m-n} \text{ for all } m. \quad (*)$$

We show that $D^{n+1}(R) * I^m \subseteq I^{m-(n+1)}$ for all m . Let δ be a differential operator on R of order $\leq n + 1$. Now

$$\delta * I^0 = \delta * R \subseteq I^{0-(n+1)} = R.$$

Using nested induction, suppose that

$$\delta * I^m \subseteq I^{m-(n+1)} \quad (**).$$

Now compute:

$$\begin{aligned} \delta * I^{m+1} &\subseteq [\delta, I] * I^m + I\delta * I^m \\ &\subseteq I^{m-n} + I(I^{m-(n+1)}) \quad (\text{Use } (*) \text{ and } (**)) \\ &= I^{m-n} = I^{(m+1)-(n+1)}. \end{aligned}$$

This completes the nested induction and shows that $\delta * I^m \subseteq I^{m-n}$ for all $\delta \in D^n(R)$ and all $m \geq 0$. \square

Remark 2.1.6. This theorem implies that $\delta \in D^n(R)$ is continuous in the I -adic topology. To see this, note that given any $a \in R$ and any basic open neighborhood $\delta(a) + I^k$ of the image $\delta(a)$, the open neighborhood $a + I^\ell$ maps into $\delta(a) + I^k$ as long as $\ell \geq k + n$.

The theorem can be easily extended to show that for R -modules M and N , every differential operator in $D_{R/k}(M, N)$ is continuous for the I -adic topologies on M and N . The proof is analogous to the case $M = N = R$ treated above.

If R is an algebra of finite type over an algebraically closed field k , then the differential operators on R are precisely those endomorphisms that are uniformly continuous for every I -adic topology on R , in the sense of the following theorem.

Theorem 2.1.7. *Let R be an algebra of finite type over an algebraically closed field k . If $\theta \in \text{End}_k(R)$ satisfies*

$$\theta(I^{n+1}) \subseteq I$$

for all $n \in \mathbb{Z}$ and ideals I ($I^n = R$ if $n \leq 0$), then $\theta \in D^n(R/k)$. Moreover, it suffices to check the condition on all maximal ideals.

Proof. The proof uses induction on n . Note that the result is clear for $n \leq -1$, since

$$\theta(m^{n+1}) = \theta(R) \subseteq m \text{ for all maximal ideals } m,$$

implies that

$$\theta(R) \in \cap \{m : m \text{ a maximal ideal}\},$$

and the right hand side of this last equation is the zero ideal by the Nullstellensatz. So $\theta = 0 \in D^n(R/k)$.

Now suppose that if $\theta(m^{n+1}) \subseteq m$ for all maximal ideals m , then $\theta \in D^n(R/k)$. Let θ be an endomorphism of R such that $\theta(m^{n+2}) \subseteq m$ for all maximal ideals m . Write R as the quotient of a polynomial ring $k[x_1, \dots, x_n]$

and let a be an element of R . Because every maximal ideal of R has the form $(x_1 - c_1, \dots, x_n - c_n)$ ($c_i \in k$), we can write $a = b + g$ where $b \in k$ and $g \in m$. Now

$$[\theta, a] = [\theta, b + g] = [\theta, b] + [\theta, g] = [\theta, g],$$

so $[\theta, R] = [\theta, m]$. The following computation uses the inductive hypothesis to show that $[\theta, a] \in D^n(R/k)$ (so $\theta \in D^{n+1}(R/k)$):

$$\begin{aligned} [\theta, a] * m^{n+1} &= [\theta, g] * m^{n+1} \\ &= \theta * (gm^{n+1}) - g\theta(m^{n+1}) \\ &\subseteq m. \end{aligned}$$

□

2.1.4 A concrete approach

In this section, let X be an affine algebraic variety defined over a field k of characteristic zero. Let R be the coordinate ring of X : $X = \text{Spec}(R)$. We write $X = \mathbb{V}(I) \subset k^N$ and $R = \frac{k[x_1, \dots, x_N]}{I}$.

Just as we define the ring of functions on X by restricting functions on the ambient space to X , we would also like to realize differential operators on X by restricting differential operators on the ambient space to X .

It is easy to see that the ring of differential operators on k^N is generated as a $k[x_1, \dots, x_N]$ -algebra by the tangents $\frac{\partial}{\partial x_i}$ (McConnell and Robson [33, 15.1.5] contains a proof). Then the ring of differential operators on k^N (or on the coordinate ring of k^N) is

$$D(k^N) = D(k[x_1, \dots, x_N]) = k[x_1, \dots, x_N, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_N}].$$

The ring of differential operators on k^N is called the Weyl algebra and is denoted W .

We want to restrict these operators (the elements of the Weyl algebra) to X (or more precisely, to act on R); however, not all of them act in a well-defined way. Only the operators in the idealizer $\mathbb{I}(I) = \{\theta \in W : \theta(I) \subseteq I\}$ act on R . Among these operators, we want to identify those operators whose image is in I with the zero operator on R . It is easy to see that these are just the operators in IW . This motivates the following description of differential operators on subvarieties of affine space.

Theorem 2.1.8. *The ring of differential operators on $X = \text{Spec}(\frac{k[x_1, \dots, x_N]}{I})$, is*

$$\frac{\{\theta \in W : \theta(I) \subseteq I\}}{IW}.$$

Proof. McConnell and Robson [33, 15.5.13] show that

$$D\left(\frac{k[x_1, \dots, x_n]}{I}\right) = \frac{\{\theta \in W : \theta(IW) \subseteq IW\}}{IW}.$$

We show that $\theta(IW) \subseteq IW$ if and only if $\theta(I) \subseteq I$. Suppose that $\theta(IW) \subseteq IW$. Then $\theta \circ I \subseteq IW$ so

$$\theta(I) = (\theta \circ I) * 1 \subseteq IW * 1 = I.$$

Conversely, suppose that $\theta(I) \subseteq I$. Let γ be an element of IW and write $\theta \circ \gamma = \sum_{\mathbf{a}} P_{\mathbf{a}} \partial^{\mathbf{a}}$. Here, $\partial^{\mathbf{a}} = \frac{1}{a_1!} \frac{\partial^{a_1}}{\partial x_1^{a_1}} \cdots \frac{1}{a_N!} \frac{\partial^{a_N}}{\partial x_N^{a_N}}$. If each $P_{\mathbf{a}} \in I$, then $\theta \circ \gamma \in IW$ and we are done. Aiming for a contradiction, assume that some $P_{\mathbf{a}} \notin I$. Let \mathbf{b} be an n -tuple of minimal total degree such that $P_{\mathbf{b}} \notin I$. Then

$$(\theta \circ \gamma) * x^{\mathbf{b}} \equiv P_{\mathbf{b}} \pmod{I}.$$

Since $\theta(I) \subseteq I$ and the image of γ is in I , it follows that the image of $\theta \circ \gamma$ is contained in I . So $P_{\mathbf{b}} \in I$, producing a contradiction. \square

2.1.5 Differential operators in prime characteristic

A bit more care must be taken with the Weyl algebra in prime characteristic since operators of the form $\frac{\partial}{\partial x}$ can be nilpotent as elements of the ring $D(R)$. For example, consider the ring $R = k[x]$ where k is a field of characteristic $p > 0$. Then

$$\frac{\partial^p}{\partial x^p} x^n = 0 \quad \text{for all } n,$$

so $\frac{\partial^p}{\partial x^p} \equiv 0$. However, as in Lemma 2.1.4, $P_{k[x]/k}^p \cong \frac{k[x, dx]}{(x-dx)^{p+1}}$. So $P_{k[x]/k}^p$ is a free $k[x]$ -module with basis $1, dx, (dx)^2, \dots, (dx)^p$. The projection onto the factor indexed by $(dx)^p$ sends $(dx)^n$ to $\delta_{n,p}$ (Kronecker delta) and induces a differential operator on $k[x]$ of order p that sends x^p to 1. This forces us to enlarge our description of $D(k[x])$ to include an operator which we can formally write as $\frac{1}{p!} \frac{\partial^p}{\partial x^p}$.

Similar considerations force us to add other operators to our description of the Weyl algebra $D(k[x_1, \dots, x_n])$ in prime characteristic. By Lemma 2.1.4 the module of differentials of order $\leq n$ is a free module with generators given by the monomials of degree $\leq n$ in dx_1, \dots, dx_n . The operators $\frac{1}{n!} \frac{\partial^n}{\partial x_i^n}$ are given by composing the universal differential operator $d : R \rightarrow P_{R/k}^n$ with the projection of $P_{R/k}^n$ onto the free summand $R \cdot (dx_i)^n$ indexed by $(dx_i)^n$. When $\frac{1}{p!} \frac{\partial^p}{\partial x_i^p}$ acts on x_i^n it returns x_i^{n-p} multiplied by an integer coefficient (since the coefficient coming from $\frac{\partial^p}{\partial x_i^p}$ is divisible by $p!$, we divide to get the coefficient of x_i^{n-p} coming from $\frac{1}{p!} \frac{\partial^p}{\partial x_i^p}$). In particular,

$$\frac{1}{n!} \frac{\partial^n}{\partial x_i^n} x_j^m = \delta_{i,j} \binom{m}{n} x_j^{m-n}.$$

Then the Weyl algebra should be:

$$W = D(k^N) = k[x_1, \dots, x_N, \dots, \frac{1}{n!} \frac{\partial^n}{\partial x_1^n}, \dots, \frac{1}{n!} \frac{\partial^n}{\partial x_N^n}, \dots].$$

When the characteristic of k is zero, this agrees with our earlier definition of the Weyl algebra. The divided power operators $\frac{1}{n!} \frac{\partial^n}{\partial x^n}$ are the natural generators of the Weyl algebra in all cases (not just when the characteristic of k is p) as they are the duals (projections) of the generators $(dx_1)^{b_1} \cdots (dx_N)^{b_N}$ of the modules of differentials. In prime characteristic, if n is not a power of the characteristic p , then the operator $\frac{1}{n!} \frac{\partial^n}{\partial x^n}$ can be written as a product of divided powers whose orders are powers of p . So we only need the operators $\frac{1}{p^e!} \frac{\partial^{p^e}}{\partial x_i^{p^e}}$ to generate the Weyl algebra. If $\mathbf{b} \in \mathbb{N}^n$, write $\partial^{\mathbf{b}}$ for the operator $\frac{1}{b_1!} \frac{\partial^{b_1}}{\partial x_1^{b_1}} \cdots \frac{1}{b_N!} \frac{\partial^{b_N}}{\partial x_N^{b_N}}$.

If X is an affine subvariety of k^N , then Theorem 2.1.8 holds in prime characteristic with this new interpretation of the Weyl algebra. In fact, the proof of Theorem 2.1.8 does not depend on the characteristic. It is easy to show that $k[x_1, \dots, x_N]$ is a simple module over the Weyl algebra. In fact, more is true: the Weyl algebra is a simple ring.

Theorem 2.1.9. *The polynomial ring $R = k[x_1, \dots, x_N]$ (k a field) is a simple module over the Weyl algebra $D(R)$ and the Weyl algebra is a simple ring.*

Proof. Fix a graded monomial ordering. Given a polynomial f , let $kx^{\mathbf{a}}$ be the term of f of maximal multi-degree. Then $\frac{1}{k} \partial^{\mathbf{a}} * f = 1$, so R is a simple D -module. To show that the Weyl algebra is simple, note that

$$\begin{aligned} \left[\frac{1}{n!} \frac{\partial^n}{\partial x^n}, x \right] &= \frac{1}{n!} \frac{\partial^n}{\partial x^n} x - x \frac{1}{n!} \frac{\partial^n}{\partial x^n} \\ &= \frac{1}{n!} \left[x \frac{\partial^n}{\partial x^n} + n \frac{\partial^{n-1}}{\partial x^{n-1}} \right] - x \frac{1}{n!} \frac{\partial^n}{\partial x^n} \\ &= \frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial x^{n-1}}. \end{aligned}$$

It follows that $\left[\frac{1}{n!} \frac{\partial^n}{\partial x_i^n}, x_j \right] = \delta_{i,j} \frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial x_i^{n-1}}$. Given a nonzero element $\theta = \sum P_{\mathbf{a}} \partial^{\mathbf{a}}$ of the Weyl algebra, let $P_{\mathbf{b}} \partial^{\mathbf{b}}$ be the term of θ whose multi-degree

is maximal. Write $x^{\mathbf{b}} = x_{i_1} \cdots x_{i_t}$. Then

$$\begin{aligned} P_{\mathbf{a}} &= \sum P_{\mathbf{a}}[\partial^{\mathbf{a}}, x^{\mathbf{b}}] \\ &= [\cdots [[\theta, x_{i_1}], x_{i_2}] \cdots, x_{i_t}] \\ &= \theta x^{\mathbf{b}} - \cdots \pm x^{\mathbf{b}} \theta. \end{aligned}$$

So the two-sided ideal generated by θ contains the polynomial $P_{\mathbf{a}}$. Since R is a simple module over the Weyl algebra, the right ideal generated by $P_{\mathbf{a}}$ contains 1. It follows that the two-sided ideal generated by θ is the entire Weyl algebra and the Weyl algebra is a simple ring. \square

The operators $\frac{1}{p!} \frac{\partial^p}{\partial x^p}$ have some very nice properties. This is a direct result of the behavior of binomial coefficients modulo p . (see [11]).

Lemma 2.1.10. *If $a = \sum_{i=0}^t a_i p^i$ and $b = \sum_{i=0}^t b_i p^i$ ($0 \leq a_i, b_i < p$) are two nonnegative integers expressed in base p expansions, then*

$$\binom{a}{b} \equiv \binom{a_0}{b_0} \binom{a_1}{b_1} \cdots \binom{a_t}{b_t} \pmod{p}.$$

Here, $\binom{c}{d} = 0$ when $c < d$.

Proof. In the ring $\frac{\mathbb{Z}}{p\mathbb{Z}}[x]$, we have

$$(1+x)^a = (1+x)^{a_0} (1+x)^{a_1 p} \cdots (1+x)^{a_t p^t} = (1+x)^{a_0} (1+x^p)^{a_1} \cdots (1+x^{p^t})^{a_t}.$$

Consider the coefficient of $x^b = \prod_{i=0}^t x^{p^i b_i}$ occurring on each side of this identity. On the left hand side the coefficient is $\binom{a}{b}$ and on the right hand side it is $\binom{a_0}{b_0} \binom{a_1}{b_1} \cdots \binom{a_t}{b_t}$. Hence, these integers are equal modulo p . \square

On $R = k[x]$ (k a field of prime characteristic p), $\frac{\partial}{\partial x}$ is R^p -linear since it is a derivation which kills p -th powers. Also, Lemma 2.1.10 can be used to show that $\frac{1}{p!} \frac{\partial^p}{\partial x^p}$ is R^{p^2} -linear:

$$\begin{aligned}
\frac{1}{p!} \frac{\partial^p}{\partial x^p} (x^{n+p^2}) &= (n+p^2) \cdots (n+p^2-p+1) x^{n+p^2-p} \\
&= \binom{n+p^2}{p} x^{n+p^2-p} \\
&= (\text{coefficient of } p \text{ in } p\text{-adic expansion of } n+p^2) x^{n+p^2-p} \\
&= (\text{coefficient of } p \text{ in } p\text{-adic expansion of } n) x^{n+p^2-p} \\
&= \binom{n}{p} x^{n-p} x^{p^2} \\
&= x^{p^2} \left(\frac{1}{p!} \frac{\partial^p}{\partial x^p} (x^n) \right).
\end{aligned}$$

This suggests a structure theorem for rings of differential operators in prime characteristic.

Theorem 2.1.11. *If $X = \text{Spec}(R)$ is an affine algebraic variety defined over a field k of prime characteristic p such that $[k^{\frac{1}{p}} : k] \leq \infty$, then*

$$D(R/\mathbb{Z}) = \cup_e \text{End}_{R^{p^e}}(R).$$

In particular, the result holds when k is a perfect field and then

$$D(R/k) = \cup_e \text{End}_{R^{p^e}}(R).$$

Proof. See Yekutieli [56] or Smith [47]. □

2.2 Hasse-Schmidt derivations

The subalgebra $\text{der}(R/k)$ of $D(R/k)$ generated by the k -derivations of R is an important algebraic object. One of the reasons for this is Grothendieck's result [12, 16.11] that the ring of differential operators on a smooth variety defined over a field of characteristic zero is generated by derivations. In prime characteristic, the derivations do not even suffice to generate $D(R/k)$

for the polynomial ring $R = k[x_1, \dots, x_N]$. As we saw in the last section, we need to include divided powers on the derivations in order to generate the Weyl algebra. In general, the derivation algebra is not particularly well-behaved in prime characteristic; however, there is an analogue of the derivation algebra, the Hasse-Schmidt algebra, whose behavior is characteristic independent. In chapter 3, we extend Grothendieck's result by showing that the ring of differential operators on a smooth variety defined over any field is equal to its Hasse-Schmidt algebra. This section introduces the Hasse-Schmidt derivations and the Hasse-Schmidt algebra and develops many of their elementary properties.

A (k -linear) Hasse-Schmidt derivation of order m on R is a finite collection of $m + 1$ k -linear endomorphisms $\Delta_m = \{\delta_i\}_{i=0}^m$ such that $\delta_0 = id_R$ and for a and b in R ,

$$\delta_n(ab) = \sum_{i+j=n} \delta_i(a)\delta_j(b) \quad (n \leq m).$$

An infinite order Hasse-Schmidt derivation on R is a collection $\Delta = \{\delta_i\}_{i=0}^\infty$ of k -linear endomorphisms of R such that $\delta_0 = id_R$ and for a and b in R ,

$$\delta_n(ab) = \sum_{i+j=n} \delta_i(a)\delta_j(b).$$

When the order of a Hasse-Schmidt derivation is not stated explicitly, we assume that it is an infinite order Hasse-Schmidt derivation. In the literature [6, 39], Hasse-Schmidt derivations are sometimes called higher order operators.

Given a Hasse-Schmidt derivation $\Delta = \{\delta_i\}_{i=0}^m$ of order $m \geq m'$ ($m \leq \infty$), we define a new Hasse-Schmidt derivation $S_{m,m'}(\Delta) = \{d_j\}_{j=0}^{m'}$ of order m' by truncation:

$$d_j = \delta_j \quad (0 \leq j \leq m').$$

The maps δ_i in a Hasse-Schmidt derivation $\Delta = \{\delta_i\}$ are called the components of Δ . The components of a Hasse-Schmidt derivation are differential operators.

Lemma 2.2.1. *Let $\Delta = \{\delta_n\}$ be a k -linear Hasse-Schmidt derivation on R . The n^{th} component δ_n of Δ is a differential operator of order $\leq n$.*

Proof. We proceed by induction on n . The case $n = 0$ is clear. Suppose that $\delta_i \in D^i(R/k)$ for $0 \leq i \leq n - 1$. For a and b in R ,

$$\begin{aligned} [\delta_n, a] * b &= \delta_n(ab) - a\delta_n(b) \\ &= \sum_{i=1}^n \delta_i(a)\delta_{n-i}(b) \\ &= \left(\sum_{i=1}^n \delta_i(a)\delta_{n-i}\right) * b. \end{aligned}$$

By the induction hypothesis, the operator $[\delta_n, a] = \sum_{i=1}^n \delta_i(a)\delta_{n-i}$ is in $D^{n-1}(R/k)$ for each $a \in R$. Thus, $J\delta_n \subseteq D^{n-1}(R/k)$ and so $J^{n+1}\delta_n = 0$. This shows that $\delta_n \in D^n(R/k)$ and completes the induction. \square

The reader should use induction on n to check that $\delta_n(1) = 0$ if $n > 0$.

The Hasse-Schmidt algebra $HS(R/k)$ is the R -subalgebra of $End_k(R)$ generated by all the components of all infinite order k -linear Hasse-Schmidt derivations on R . By Lemma 2.2.1, the Hasse-Schmidt algebra $HS(R/k)$ is a subalgebra of the ring of differential operators, $D(R/k)$.

Example 2.2.2. On $R = k[x_1, \dots, x_N]$, the collection of operators $\Delta_i = \{\delta_{i,n}\}_{n=0}^{\infty}$ given by

$$\delta_{i,n} = \begin{cases} id_R, & n = 0 \\ \frac{1}{n!} \frac{\partial^n}{\partial x_i^n} & n > 0 \end{cases}$$

is a Hasse-Schmidt derivation independent of the characteristic of k . To check this, it suffices to show that

$$\delta_{i,n}(ab) = \sum_{\ell+j=n} \delta_{i,\ell}(a)\delta_{i,j}(b)$$

for monomials a and b . Furthermore, since the maps $\delta_{i,n}$ are $k[x_1, \dots, \hat{x}_i, \dots, x_N]$ -linear (here, we remove the variable x_i), it suffices to show that

$$\delta_{i,n}(x_i^c x_i^d) = \sum_{\ell+j=n} \delta_{i,\ell}(x_i^c) \delta_{i,j}(x_i^d).$$

To this end, we compute:

$$\begin{aligned} \delta_{i,n}(x_i^{c+d}) &= \frac{1}{n!} \frac{\partial^n}{\partial x_i^n} * x_i^{c+d} \\ &= \binom{c+d}{n} x_i^{c+d-n} \\ &= \left(\sum_{\ell+j=n} \binom{c}{\ell} \binom{d}{j} \right) x_i^{c+d-n} \\ &= \sum_{\ell+j=n} \binom{c}{\ell} x_i^{c-\ell} \binom{d}{j} x_i^{d-j} \\ &= \sum_{\ell+j=n} \delta_{i,\ell}(x_i^c) \delta_{i,j}(x_i^d). \end{aligned}$$

It follows that

$$k[x_1, \dots, x_N, \dots, \frac{1}{n!} \frac{\partial^n}{\partial x_1^n}, \dots, \frac{1}{n!} \frac{\partial^n}{\partial x_N^n}, \dots] \subseteq HS(R/k).$$

However,

$$HS(R/k) \subseteq D(R/k) = k[x_1, \dots, x_N, \dots, \frac{1}{n!} \frac{\partial^n}{\partial x_1^n}, \dots, \frac{1}{n!} \frac{\partial^n}{\partial x_N^n}, \dots],$$

so the Hasse-Schmidt algebra on a polynomial ring is just the Weyl algebra,

$$HS(R/k) = k[x_1, \dots, x_N, \dots, \frac{1}{n!} \frac{\partial^n}{\partial x_1^n}, \dots, \frac{1}{n!} \frac{\partial^n}{\partial x_N^n}, \dots].$$

Example 2.2.3. There are examples of derivations that do not appear as the first component of a Hasse-Schmidt derivation. Let $R = \frac{k[x,y,z]}{(xy-z^2)}$ where k is a field of characteristic 2. Following a fairly standard practice, we represent elements of R (which are equivalence classes) by an element of the class itself. Then $\frac{\partial}{\partial z}$ induces a derivation on R because it kills $xy - z^2$. However,

$$\frac{1}{2!} \frac{\partial^2}{\partial z^2} * (xy - z^2) = 1,$$

so $\frac{1}{2!} \frac{\partial^2}{\partial z^2}$ is not a differential operator on R . This suggests that $\frac{\partial}{\partial z}$ is not the first component of a Hasse-Schmidt derivation on R . Before verifying this, we note that if $\{d_n\}$ is a Hasse-Schmidt derivation with $d_1 = \frac{\partial}{\partial z}$, then

$$d_2(z^2) = zd_2(z) + d_1(z)d_1(z) + d_2(z)z = (d_1(z))^2 = 1.$$

Now we produce a contradiction by showing that d_2 does not define an operator on R . We compute,

$$\begin{aligned} d_2(xy - z^2) &= d_2(xy) - d_2(z^2) \\ &= d_2(x)y + d_1(x)d_1(y) + xd_2(y) - (d_1(z))^2 \\ &= d_2(x)y + xd_2(y) - 1 \\ &\equiv -1 \pmod{(x, y, z)}. \end{aligned}$$

Thus, $d_2(xy - z^2) \notin (xy - z^2)$ and d_2 does not act on R . It follows that no Hasse-Schmidt derivation, $\{d_n\}$ has $d_1 = \frac{\partial}{\partial z}$. A simpler example will be given in Example 4.2.7 once we develop the notion of *HS*-stability.

In contrast to the characteristic p situation, every derivation appears as the first component of some Hasse-Schmidt derivation in characteristic zero. In fact, in characteristic zero the Hasse-Schmidt algebra equals the derivation algebra, $HS(R/k) = der(R/k)$. In this sense, the Hasse-Schmidt algebra is an analogue of the derivation algebra that is better behaved in prime characteristic.

Theorem 2.2.4 (Ribenoim [39]). *If R is a k -algebra over a field k of characteristic zero, then $der(R/k) = HS(R/k)$ and for each derivation $d \in Der(R/k)$, there is a Hasse-Schmidt derivation $\Delta = \{\delta_i\}$ such that $\delta_1 = d$.*

Hasse-Schmidt derivations on R induce ring endomorphisms of the power series algebra $R[[t]]$. Given a k -algebra R and a Hasse-Schmidt derivation

$\Delta = \{\delta_n\} \subset HS(R/k)$, we form ring maps

$$e^{t\Delta} : R[[t]] \rightarrow R[[t]]$$

and

$$e_m^{t\Delta} : \frac{R[[t]]}{(t^{m+1})} \rightarrow \frac{R[[t]]}{(t^{m+1})}$$

as follows. First, extend the action of δ_n to $R[[t]]$ and $\frac{R[[t]]}{(t^{m+1})}$ by linearity: $\delta_n(at^i) = \delta_n(a)t^i$. Then set

$$e^{t\Delta} = \delta_0 + \delta_1 t + \delta_2 t^2 + \dots$$

and

$$e_m^{t\Delta} = \delta_0 + \delta_1 t + \delta_2 t^2 + \dots + \delta_m t^m.$$

Now we check that $e^{t\Delta}$ and $e_m^{t\Delta}$ are ring maps. They are additive by definition. To check that $e^{t\Delta}$ is multiplicative, it suffices to verify that $e^{t\Delta}(ab) = e^{t\Delta}(a)e^{t\Delta}(b)$ for a and b in R :

$$\begin{aligned} e^{t\Delta}(ab) &= \sum_{n=0}^{\infty} t^n \delta_n(ab) \\ &= \sum_{n=0}^{\infty} t^n \sum_{i+j=n} \delta_i(a) \delta_j(b) \\ &= \sum_{n=0}^{\infty} \sum_{i+j=n} t^i \delta_i(a) t^j \delta_j(b) \\ &= [\sum_{i=0}^{\infty} t^i \delta_i(a)] [\sum_{j=0}^{\infty} t^j \delta_j(b)] \\ &= e^{t\Delta}(a) e^{t\Delta}(b). \end{aligned}$$

This also shows that $e_m^{t\Delta}$ is multiplicative since $e^{t\Delta} \equiv e_m^{t\Delta} \pmod{t^{m+1}}$.

We use this observation to show that in prime characteristic p , the action of Hasse-Schmidt derivations on p^e -th powers satisfies an interesting relation.

Lemma 2.2.5. *If the characteristic of R is p and $\Delta = \{\delta_n\}$ is a Hasse-Schmidt derivation, then for $g \in R$,*

$$\delta_n(g^{p^e}) = \begin{cases} 0 & \text{if } p^e \text{ does not divide } n, \\ (\delta_{\frac{n}{p^e}}(g))^{p^e} & \text{if } p^e \text{ divides } n. \end{cases}$$

Proof. This is an immediate consequence of the following computation in $R[[t]]$ (equate coefficients of t).

$$\begin{aligned}
g^{p^e} + (\delta_1(g)t)^{p^e} + (\delta_2(g)t^2)^{p^e} + \cdots &= (g + \delta_1(g)t + \delta_2(g)t^2 + \cdots)^{p^e} \\
&= [e^{t\Delta}(g)]^{p^e} \\
&= e^{t\Delta}(g) \cdots e^{t\Delta}(g) \quad (p^e \text{ copies}) \\
&= e^{t\Delta}(g^{p^e}) \\
&= g^{p^e} + \delta_1(g^{p^e})t + \delta_2(g^{p^e})t^2 + \cdots
\end{aligned}$$

□

Note that $e^{t\Delta}(t) = t$. This fact helps to characterize those endomorphisms ϕ of $R[[t]]$ that arise as maps of the form $e^{t\Delta}$. In fact, we can identify Hasse-Schmidt derivations with certain ring endomorphisms of $R[[t]]$.

Theorem 2.2.6. *Let $\frac{R[[t]]}{(t^{m+1})} \xrightarrow{\mu_m} \frac{R[[t]]}{(t)} \cong R$ and $R[[t]] \xrightarrow{\mu} \frac{R[[t]]}{(t)} \cong R$ be the quotient maps. If ϕ_m is a ring endomorphism of $\frac{R[[t]]}{(t^{m+1})}$ such that $\phi_m(t) = t$ and $\mu_m \circ \phi_m$ induces the identity map on R , then $\phi_m = e_m^{t\Delta}$ for some Hasse-Schmidt derivation Δ of order m . Similarly, if ϕ is a ring endomorphism of $R[[t]]$ such that $\phi(t) = t$ and $\mu \circ \phi$ induces the identity map on R , then $\phi = e^{t\Delta}$ for some Hasse-Schmidt derivation Δ .*

Proof. Suppose ϕ_m is a ring endomorphism of $\frac{R[[t]]}{(t^{m+1})}$ such that $\phi_m(t) = t$ and $(\mu_m \circ \phi_m)|_R = id_R$. For $a \in R$, write

$$\phi_m(a) = d_0(a) + d_1(a)t + \cdots + d_m(a)t^m.$$

This defines additive maps $d_i : R \rightarrow R$. The map d_0 is the identity map: $d_0(a) = (\mu_m \circ \phi_m)|_R(a) = a$. Because ϕ_m is a ring map:

$$\sum_{i=0}^m d_i(ab)t^i = \phi_m(ab) = \phi_m(a)\phi_m(b) = \sum_{i=0}^m \left(\sum_{j+k=i} d_j(a)d_k(b) \right) t^i,$$

for $a, b \in R$. So $d_i(ab) = \sum_{j+k=i} d_j(a)d_k(b)$ and the maps $\Delta_m = \{d_i\}$ form a Hasse-Schmidt derivation of order $\leq m$. Because ϕ_m and $e_m^{t\Delta}$ are ring endomorphisms of $\frac{R[[t]]}{(t^{m+1})}$ that agree on R and t , they agree everywhere: $\phi_m = e_m^{t\Delta}$.

A similar argument establishes the second claim. Alternatively, the endomorphism ϕ of $R[[t]]$ induces endomorphisms ϕ_m of $\frac{R[[t]]}{(t^{m+1})}$. These give rise to Hasse-Schmidt derivations Δ_m and these glue together: $S_{m',m}(\Delta_{m'}) = \Delta_m$ for $m' \geq m$. In the limit, we obtain a Hasse-Schmidt derivation $\Delta = \{d_i\}_{i=0}^{\infty}$ of R such that $e^{t\Delta}$ agrees with ϕ on every polynomial. Since both maps preserve t -adic order, $\phi = e^{t\Delta}$. \square

The Hasse-Schmidt derivations form a group under the convolution product. If $\Delta = \{\delta_i\}$ and $\Gamma = \{\gamma_j\}$ are Hasse-Schmidt derivations then this product is defined to be the Hasse-Schmidt derivation corresponding to $e^{t\Delta} \circ e^{t\Gamma}$. We compute,

$$\begin{aligned} e^{t\Delta} \circ e^{t\Gamma} &= \left(\sum_{i=0}^{\infty} t^i \delta_i \right) \left(\sum_{j=0}^{\infty} t^j \gamma_j \right) \\ &= \sum_{k=0}^{\infty} t^k \left(\sum_{i+j=k} \delta_i \circ \gamma_j \right) \\ &= e^{t(\Delta * \Gamma)}. \end{aligned}$$

Thus, the product of $\Delta = \{\delta_i\}$ and $\Gamma = \{\gamma_j\}$ is the Hasse-Schmidt derivation $\Delta * \Gamma = \{\eta_k\}$ where

$$\eta_k = \sum_{i+j=k} \delta_i \circ \gamma_j.$$

The identity element is the trivial Hasse-Schmidt derivation, $E = \{\epsilon_i\}$, where

$$\epsilon_i = \begin{cases} id_R & i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The identity E corresponds to the map $e^{tE} = id_{R[[t]]}$. We refer to [13] for the

proof that the Hasse-Schmidt derivations form a group under this product (the non-trivial fact is that inverses exist).

Lemma 2.2.7. *If Δ is a Hasse-Schmidt derivation on R with inverse Δ' , then the endomorphisms $e^{t\Delta}$ and $e^{t\Delta'}$ of $R[[t]]$ are mutually inverse ring automorphisms of $R[[t]]$.*

Proof. This is obvious since:

$$e^{t\Delta} \circ e^{t\Delta'} = e^{t(\Delta*\Delta')} = e^{tE} = id_{R[[t]]} = e^{t(\Delta'*\Delta)} = e^{t\Delta'} \circ e^{t\Delta}.$$

□

Chapter 3

Differential Operators on Étale Extensions

The aim of this chapter is to show that if R is smooth over a field k , then $D(R/k) = HS(R/k)$ and R is a product of D-simple domains. Inspired by Grothendieck's treatment of differential operators in EGA [12], we prove these results in a characteristic independent manner by considering the behavior of differential operators on étale extensions. Our approach to differential operators and étale maps mimics that of Msson [30].

We also show that Hasse-Schmidt derivations extend over formally étale extensions and settle a question due to Brown and Kuan.

3.1 Étale extensions

This section collects some definitions and standard results about étale and smooth extensions that will be used later. Motivated by pragmatic considerations, our account is extremely terse. For more details we refer the reader

to Grothendieck's very detailed account in EGA [12, chapters 17 and 18; in French] or Milne [36, in English].

Definition 3.1.1. Let $A \xrightarrow{f} B$ be a homomorphism of commutative rings. The map f is formally smooth if the following condition holds. For all commutative diagrams

$$\begin{array}{ccc} B & \xrightarrow{u} & C/J \\ \uparrow f & & \uparrow \\ A & \xrightarrow{v} & C \end{array}$$

where C is a commutative ring, $J \subset C$ is a nilpotent ideal and u, v are ring homomorphisms, there is a ring homomorphism $u' : B \rightarrow C$ lifting u ; that is, u' completes the diagram

$$\begin{array}{ccc} B & \xrightarrow{u} & C/J \\ \uparrow f & \searrow u' & \uparrow \\ A & \xrightarrow{v} & C \end{array}$$

The map f is formally étale if and only if there is one and only one such u' .

Remark 3.1.2. If A, B and C are all algebras over a commutative ring k and the maps f, u and v are all k -algebra homomorphisms, then the lifting u' of u is also a k -algebra homomorphism.

Definition 3.1.3. The ring map $A \xrightarrow{f} B$ is smooth (respectively, étale) if B is an A -algebra of finite type and f is formally smooth (respectively, formally étale).

Remark 3.1.4. The reader is warned that there are several different conventions in use in the literature regarding the terms smooth and formally smooth. Matsumura [31] speaks about formally smooth maps with reference to topologies on the rings. Our definition of formally smooth is equivalent to Matsumura’s definition of smoothness: a formally smooth map with respect to the discrete topologies. Our notion of smooth is sometimes referred to as smooth of finite type.

Lemma 3.1.5. *If $A \xrightarrow{f} B$ is formally smooth (respectively, formally étale) and S and T are multiplicatively closed subsets of A and B such that $f(S) \subseteq T$, then the corresponding ring map $S^{-1}A \rightarrow T^{-1}B$ is also formally smooth (respectively, formally étale). In particular, if S is a multiplicatively closed subset of A , then the map $A \rightarrow S^{-1}A$ is formally étale.*

Proof. Suppose that $A \xrightarrow{f} B$ is formally étale. Let C be a $T^{-1}B$ -algebra and $J \subset C$ be a nilpotent ideal. Suppose that we have the following diagram of maps:

$$\begin{array}{ccc} T^{-1}B & \longrightarrow & C/J \\ \uparrow & & \uparrow \\ S^{-1}A & \longrightarrow & C \end{array}$$

This induces the following diagram:

$$\begin{array}{ccccc} B & \longrightarrow & T^{-1}B & \longrightarrow & C/J \\ \uparrow & & \uparrow & & \uparrow \\ f & & & & \\ A & \longrightarrow & S^{-1}A & \longrightarrow & C \end{array}$$

Since $A \xrightarrow{f} B$ is formally étale, there is a unique ring homomorphism $B \xrightarrow{u} C$ fitting into the diagram above. Since the map $B \rightarrow C/J$ factors through $T^{-1}B$, the image of T under this map lies in the set of invertible elements of C/J . Since J is a nilpotent ideal, every lifting of one of these elements is an invertible element of C . It follows that u induces a ring homomorphism $T^{-1}B \xrightarrow{u'} C$ making the first diagram commute. Because the map u is unique, so is the map u' . So the extension $S^{-1}A \rightarrow T^{-1}B$ is also formally étale. The other assertions are proven in a similar way. \square

Lemma 3.1.6. *Formally smooth morphisms are flat.*

Proof. See Milne [36, Proposition 3.24]. \square

Lemma 3.1.7. *If $A \rightarrow B$ is étale, then $\Omega_{B/A}^1 = 0$.*

Proof. See Milne [36, Proposition 3.5]. \square

Lemma 3.1.8. *If $A \rightarrow B$ is smooth and A is regular, then B is also regular.*

Proof. See Milne [36, Proposition 3.17]. \square

Smooth morphisms over a field factor locally through a formally étale morphism.

Theorem 3.1.9 (Grothendieck [12, 17.15.9]). *Let R be a smooth algebra of finite type over a field k and let m be a maximal ideal of R . Then there exists a regular system of parameters x_1, \dots, x_n for R_m such that the map $k \rightarrow R_m$ factors as*

$$k \hookrightarrow k[x_1, \dots, x_n] \hookrightarrow R_m,$$

where $k[x_1, \dots, x_n] \hookrightarrow R_m$ is formally étale. Note that the elements x_1, \dots, x_n are algebraically independent so that $k[x_1, \dots, x_n]$ is a polynomial ring.

3.2 Differential operators on étale extensions

Rings of differential operators are particularly well-behaved for étale extensions.

Proposition 3.2.1. *Let $A \rightarrow B$ be an étale extension of rings. Then $D(B/A) = B$.*

Proof. Since $A \rightarrow B$ is étale, $\Omega_{B/A}^1 = 0$. Then $\text{Der}_A(B) = \text{Hom}_A(\Omega_{B/A}^1, B) = 0$ and $D^1(B/A) = B \oplus \text{Der}_A(B) = B$. Now we show that $D^n(B/A) = D^{n-1}(B/A)$. Let J be the kernel of the multiplication map $B \otimes_A B \rightarrow B$ and recall that $D^n(B/A) = \{\theta \in \text{End}_A(B) : J^{n+1}\theta = 0\}$. Let $\theta \in D^n(B/A)$. Then $J^{n-1}\theta \in D^1(B/A) = B$, so $J^n\theta = J(J^{n-1}\theta) = 0$. Then $\theta \in D^{n-1}(B/A)$. So $D^n(B/A) = D^{n-1}(B/A)$, and by recursion, we get $D^n(B/A) = B$. Since this holds for all n , $D(B/A) = \cup_n D_n(B/A) = B$. \square

This suggests that the rings $D(A/k)$ and $D(B/k)$ may be related. Pursuing this idea, we first study the relation between $P_{A/k}^n$ and $P_{B/k}^n$.

Lemma 3.2.2. *If $A \xrightarrow{f} B$ is a formally étale extension of k -algebras, then there is a unique differential operator $\tilde{d} \in D_B^n(B, B \otimes_A P_A^n)$ of order $\leq n$*

making the following diagram commute.

$$\begin{array}{ccc}
 B & \xrightarrow{1} & B \\
 \uparrow f & \searrow \tilde{d} & \uparrow id_B \otimes \mu_A \\
 A & \xrightarrow{f \otimes d_A^n} & B \otimes P_A^n
 \end{array}$$

Here the map $\mu_A : P_A^n \rightarrow A$ is the map $P_A^n \rightarrow \frac{P_A^n}{J} \cong A$.

Proof. Because the map f is formally étale, it is also flat (Lemma 3.1.6). So the kernel K of $id_B \otimes \mu_A$ is just $B \otimes \frac{J}{J^{n+1}}$. Then K is nilpotent and $K^{n+1} = 0$. Since f is formally étale, there is a unique ring homomorphism \tilde{d} making the diagram commute.

It remains to show that \tilde{d} is a differential operator of order $\leq n$. Since \tilde{d} is a ring homomorphism,

$$[\tilde{d}, b] * \gamma = (\tilde{d} * b)(\tilde{d} * \gamma) - b\tilde{d} * \gamma = (\tilde{d} * b - b \otimes 1)\tilde{d} * \gamma,$$

for all $b, \gamma \in R$. Thus, $[\tilde{d}, b] = (\tilde{d} * b - b \otimes 1)\tilde{d}$ for all $b \in R$. By induction, we have

$$[\cdots [[\tilde{d}, b_0], b_1] \cdots], b_n = (\tilde{d} * b_0 - b_0 \otimes 1) \cdots (\tilde{d} * b_n - b_n \otimes 1)\tilde{d}.$$

By the commutativity of the top triangle in the diagram, $\tilde{d} * b - b \otimes 1 \in K$ for all $b \in B$. Thus, $[\cdots [[\tilde{d}, b_0], b_1] \cdots], b_n \in K^{n+1} = (0)$ for all $b_0, \dots, b_n \in B$ and \tilde{d} is a differential operator from B to the B -module $B \otimes_A P_A^n$ of order $\leq n$. \square

Theorem 3.2.3. *When $A \xrightarrow{f} B$ is a formally étale extension of k -algebras, the pair $(\tilde{d}, B \otimes_A P_A^n)$ satisfies the same universal mapping property as (d_B^n, P_B^n) . In particular, the B -modules $B \otimes_A P_A^n$ and P_B^n are canonically*

isomorphic via a B -module map ϕ_B making the following diagram commute.

$$\begin{array}{ccc}
 B & \xrightarrow{\tilde{d}} & B \otimes_A P_A^n \\
 & \searrow^{d_B^n} & \downarrow \phi_B \\
 & & P_B^n
 \end{array}$$

Proof. It is sufficient to show that there is a B -module homomorphism $B \otimes_A P_A^n \xrightarrow{\phi_B} P_B^n$ such that $\phi_B \circ \tilde{d} = d_B^n$. To see that this is sufficient, let $B \xrightarrow{D} B$ be a differential operator of order $\leq n$. There is a unique B -module homomorphism $P_B^n \xrightarrow{\gamma_D} B$ such that $\gamma_D \circ d_B^n = D$. Composing with ϕ_B , we get a map $B \otimes_A P_A^n \xrightarrow{\gamma_D \circ \phi_B} B$ such that $\gamma_D \circ \phi_B \circ \tilde{d} = D$.

$$\begin{array}{ccccc}
 & & B & & \\
 & \swarrow D & \downarrow d_B^n & \searrow \tilde{d} & \\
 B & \xleftarrow{\gamma_D} & P_B^n & \xleftarrow{\phi_B} & B \otimes_A P_A^n
 \end{array}$$

Now we show that the map $\gamma_D \circ \phi_B$ is the unique B -module homomorphism $B \otimes_A P_A^n \xrightarrow{\psi} B$ such that $\psi \circ \tilde{d} = D$. To this end, suppose that ψ is such a homomorphism:

$$\begin{array}{ccc}
 B & \xrightarrow{\tilde{d}} & B \otimes_A P_A^n \\
 & \searrow D & \downarrow \psi \\
 & & B
 \end{array}$$

Because P_A^n is generated as an A -module by the image of d_A^n , $B \otimes_A P_A^n$ is generated as a B -module by the image of $A \xrightarrow{1 \otimes d_A^n} B \otimes_A P_A^n$. Since $\tilde{d} \circ f =$

$1 \otimes d_A^n, B \otimes_A P_A^n$ is generated by the image of \tilde{d} . Since both the B -module homomorphisms ψ and $\gamma_D \circ \phi_B$ agree on the image of \tilde{d} , they must agree on all of $B \otimes_A P_A^n$: $\psi = \gamma_D \circ \phi_B$. Thus, $(\tilde{d}, B \otimes_A P_A^n)$ satisfies the same universal mapping property as (d_B^n, P_B^n) . Using the universal mapping properties, we see that if the homomorphism ϕ_B exists, then it is an isomorphism.

The composition $d_B^n \circ f : A \xrightarrow{f} B \xrightarrow{d_B^n} P_B^n$ is a differential operator in $D_{A/k}(A, B \otimes_A P_A^n)$. So by the universal property of (d_A^n, P_A^n) , there is a unique A -module homomorphism $P_A^n \xrightarrow{\phi_A} P_B^n$ such that $\phi_A \circ d_A^n = d_B^n \circ f$. Because Hom and \otimes are adjoint functors, there is a unique B -module homomorphism $B \otimes_A P_A^n \xrightarrow{\phi_B} P_B^n$ such that $\phi_B \circ (1 \otimes id) = \phi_A$.

$$\begin{array}{ccc} P_A^n & \xrightarrow{1 \otimes id} & B \otimes_A P_A^n \\ & \searrow \phi_A & \downarrow \phi_B \\ & & P_B^n \end{array}$$

We claim that $\phi_B \circ \tilde{d} = d_B^n$. Consider the following diagrams.

$$\begin{array}{ccc} B & \xrightarrow{1} & B \\ \uparrow f & \searrow \phi_B \circ \tilde{d} & \uparrow \mu_B \\ A & \xrightarrow{d_B^n \circ f} & P_B^n \end{array} \quad \text{and} \quad \begin{array}{ccc} B & \xrightarrow{1} & B \\ \uparrow f & \searrow d_B^n & \uparrow \mu_B \\ A & \xrightarrow{d_B^n \circ f} & P_B^n \end{array}$$

In these diagrams, $\mu_B : P_B^n \rightarrow B$ is the map $P_B^n \rightarrow \frac{P_B^n}{J_B} \cong B$.

The second diagram is clearly commutative. Because $A \xrightarrow{f} B$ is formally étale, the lifting of $B \xrightarrow{1} B$ to $B \rightarrow P_B^n$ is unique. If the first diagram is commutative, this guarantees that $\phi_B \circ \tilde{d} = d_B^n$.

Now we check that the first diagram is commutative. We start with the lower triangle. For $a \in A$ we have

$$(\phi_B \circ \tilde{d} \circ f)(a) = \phi_B(\tilde{d}(f(a))) = \phi_B(1 \otimes d_A^n(a)) = \phi_A(d_A^n(a)) = d_B^n(f(a)) = (d_B^n \circ f)(a).$$

For the upper triangle, note that

$$(\mu_B \circ \phi_B)(1 \otimes d_A^n(a)) = \mu_B(d_B^n(f(a))) = f(a) = (id_B \otimes \mu_A)(1 \otimes d_A^n(a)),$$

so $\mu_B \circ \phi_B$ and $id_B \otimes \mu_A$ agree on tensors of the form $1 \otimes d_A^n(a) \in B \otimes_A P_A^n$. Because both the maps $\mu_B \circ \phi_B$ and $id_B \otimes \mu_A$ are B -module maps and $B \otimes P_A^n$ is generated as a B -module by tensors of the form $1 \otimes d_A^n(a)$ ($a \in A$), $\mu_B \circ \phi_B = id_B \otimes \mu_A$. Now it is easy to see that the top triangle also commutes:

$$\mu_B \circ (\phi_B \circ \tilde{d}) = (id_B \otimes \mu_A) \circ \tilde{d} = id_B.$$

The map ϕ_B satisfies $\phi_B \circ \tilde{d} = d_B^n$. □

Now we can show that under very mild hypotheses on an étale extension $A \rightarrow B$ of k -algebras, $D(B/k) \cong D(A/k) \otimes_A B$. Recall that an R -module M is finitely presented if there is an exact sequence

$$F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

where F_0 and F_1 are free R -modules. It is immediate that any finitely generated module over a Noetherian ring is finitely presented. In particular, if A is an algebra of finite type over a field k , then $P_{A/k}^n$ is a finitely presented A -module for all n . If $B = S^{-1}A$, then $P_{B/k}^n = P_{A/k}^n \otimes_A B$ because Theorem 3.2.3 can be applied to the formally étale localization map. So $P_{B/k}^n$ is finitely presented over B . Finitely presented modules have some particularly nice homological properties. For example, we have the following well-known theorem.

Lemma 3.2.4. *Let $R \rightarrow B$ be a flat map of rings and let N be a finitely presented R -module. Then for any R -module M , there is a natural isomorphism*

$$\mathrm{Hom}_R(N, M) \otimes_R B \rightarrow \mathrm{Hom}_B(N \otimes_R B, M \otimes_R B)$$

given by $g \otimes b \mapsto g_b$, where $g_b(n \otimes b') = g(n) \otimes bb'$.

Proof. Since N is finitely presented, there is an exact sequence

$$F_1 \rightarrow F_0 \rightarrow N \rightarrow 0,$$

where the F_0 and F_1 are finitely generated free R -modules. Apply the two functors $\mathrm{Hom}_R(_, M) \otimes_R B$ and $\mathrm{Hom}_B(_ \otimes_R B, M \otimes_R B)$ and use the fact that B is flat to obtain the sequences

$$\begin{array}{ccccc} \mathrm{Hom}_R(N, M) \otimes_R B & \longrightarrow & \mathrm{Hom}_R(F_0, M) \otimes_R B & \longrightarrow & \mathrm{Hom}_R(F_1, M) \otimes_R B \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}_B(N \otimes_R B, M \otimes_R B) & \longrightarrow & \mathrm{Hom}_B(F_0 \otimes_R B, M \otimes_R B) & \longrightarrow & \mathrm{Hom}_B(F_1 \otimes_R B, M \otimes_R B). \end{array}$$

Because the two arrows on the right are isomorphisms, so is the arrow on the left. \square

Theorem 3.2.5. *If A is an algebra essentially of finite type over a field k and $A \rightarrow B$ is a formally étale extension of k -algebras, then $D^n(B/k)$ is canonically isomorphic to $D^n(A/k) \otimes_A B$. Furthermore, $D(B/k) \cong D(A/k) \otimes_A B$.*

Proof. By Theorem 3.2.3,

$$D^n(B/k) \cong \mathrm{Hom}_B(F_{B/k}^n, B) \cong \mathrm{Hom}_B(F_{A/k}^n \otimes_A B, B).$$

Also, we can apply Theorem 3.2.4 since $P_{A/k}^n$ is finitely presented (as A is essentially of finite type over a field). This gives,

$$\text{Hom}_B(P_{A/k}^n \otimes_A B, B) \cong \text{Hom}_A(P_{A/k}^n, A) \otimes_A B \cong D^n(A/k) \otimes_A B.$$

Thus, $D^n(B/k) \cong D^n(A/k) \otimes_A B$. This implies the last assertion because direct limits commute with tensor products. \square

We end this section by examining the behavior of Hasse-Schmidt derivations over formally étale extensions.

Theorem 3.2.6. *Let $A \xrightarrow{f} B$ be a formally étale extension of k -algebras and suppose that $\Delta = \{\delta_n\} \subset HS(A/k)$ is a Hasse-Schmidt derivation on A . Then there exists a unique Hasse-Schmidt derivation $\gamma = \{\gamma_n\}$ on B such that $f \circ \delta_n = \gamma_n \circ f$.*

Proof. Fix a non-negative integer m . Let $A \xrightarrow{\iota} \frac{A[[t]]}{(t^{m+1})}$ be the natural inclusion, $\iota(a) = a$, and note that $A \xrightarrow{f} B$ extends to a map $\frac{A[[t]]}{(t^{m+1})} \xrightarrow{\hat{f}} \frac{B[[t]]}{(t^{m+1})}$. Let $\frac{B[[t]]}{(t^{m+1})} \xrightarrow{g} B$ be the quotient by $tB[[t]]$. Then we have the map $\hat{f} \circ e_m^\Delta \circ \iota$ and the following diagram.

$$\begin{array}{ccc} B & \xrightarrow{id_B} & B \\ \uparrow f & \searrow e_m^\Gamma & \uparrow g \\ A & \xrightarrow{\hat{f} \circ e_m^\Delta \circ \iota} & \frac{B[[t]]}{(t^{m+1})} \end{array}$$

Here, the ring map e_m^Γ exists because B is formally smooth over A . Forcing e_m^Γ to commute with multiplication by t , we extend e_m^Γ to a ring map $e_m^\Gamma : \frac{B[[t]]}{(t^{m+1})} \rightarrow \frac{B[[t]]}{(t^{m+1})}$. Furthermore,

$$e_m^\Gamma(t) = e_m^\Gamma(1)t = e_m^\Delta(1)t = t$$

and $(g \circ e_m^\Gamma)|_B = id_B$. By Theorem 2.2.6, e_m^Γ gives rise to a Hasse-Schmidt derivation $\gamma_m = \{\gamma_n\}_{n=0}^m$ of order m that extends $S_{\infty,m}(\Delta)$. Because $A \xrightarrow{f} B$ is formally étale, the maps e_m^Γ are unique. This forces $S_{m,m'}(\gamma_m) = \gamma_{m'}$ for $m' < m$. The Hasse-Schmidt derivations γ_m give rise to a limiting Hasse-Schmidt derivation $\gamma = \{\gamma_n\}_{n=0}^\infty$ that extends Δ . Because the liftings e_m^Γ are unique, so is γ . \square

This extends a result observed by both Brown [5] and Ribenboim [40]: Hasse-Schmidt derivations extend to localizations.

Corollary 3.2.7. *If $\Delta = \{\delta_n\}$ is a Hasse-Schmidt derivation on A over k and T is a multiplicative subset of A , then there is a unique Hasse-Schmidt derivation on $T^{-1}A$ extending Δ .*

Proof. This follows immediately from Theorem 3.2.6 because the localization map $A \rightarrow T^{-1}A$ is formally étale. \square

Brown and Kuan [7, page 405] raise the question of whether every Hasse-Schmidt derivation on a localization $T^{-1}A$ is extended from A . This is not the case, as the following example shows.

Example 3.2.8. Hasse-Schmidt derivations on $k[x, y, \frac{1}{y}]$ correspond to ring endomorphisms of $k[x, y, \frac{1}{y}][[t]]$ that send t to t and project to the identity on $k[x, y, \frac{1}{y}]$. Such an endomorphism θ is completely determined by the image of x and y . Let

$$\begin{aligned}\theta(x) &= x + d_1(x)t + d_2(x)t^2 + \cdots & \text{and} \\ \theta(y) &= y + d_1(y)t + d_2(y)t^2 + \cdots\end{aligned}$$

Because x and y are algebraically independent in $k[x, y, \frac{1}{y}]$, we are free to choose any values for $d_i(x), d_j(y)$ in $k[x, y, \frac{1}{y}]$. Set $d_j(y) = 0$ and $d_n(x) =$

y^{-n} . The map θ is just $e^{t\Delta}$ where $\Delta = \{d_n\}$ is a Hasse-Schmidt derivation on $k[x, y, \frac{1}{y}]$. Such a Hasse-Schmidt derivation is the extension of a Hasse-Schmidt derivation $\gamma = \{\gamma_n\}$ on $k[x, y]$ if there exists a $g \in k[x, y]$ such that $\theta = g\Delta = \{g^n d_n\}$. Since the powers of y required to rationalize d_n grow exponentially, no such element g exists.

3.3 Differential operators on smooth extensions

The results concerning differential operators on étale extensions can be applied to yield information regarding differential operators on smooth extensions in arbitrary characteristic. In characteristic zero, this material is well-known, though our approach differs from the standard treatment (see [33]).

Theorem 3.3.1. *Let R be a smooth algebra of finite type over a field k . Then the ring of differential operators equals the Hasse-Schmidt algebra $D(R/k) = HS(R/k)$.*

Proof. Let m be a maximal ideal of R . By Theorem 3.1.9, there exists a regular system of parameters $x_1, \dots, x_n \in R_m$ giving rise to the following diagram.

$$\begin{array}{ccc}
 & A = k[x_1, \dots, x_n] & \\
 & \nearrow & \downarrow F \\
 k & \longrightarrow & R_m
 \end{array}$$

where the injection F is formally étale and the ring A is a polynomial ring with coefficients in k . From Theorem 3.2.5,

$$D(R_m/k) \cong D(A/k) \otimes_A R_m. \quad (*)$$

In example 2.2.2 we saw that

$$D(A/k) = HS(A/k).$$

By Theorem 3.2.6, we can lift each Hasse-Schmidt derivation on A to a Hasse-Schmidt derivation on R_m . Because the Hasse-Schmidt derivations on A generate $D(A/k)$, their liftings over F generate the R_m -algebra $D(R_m)$ (using $(*)$). So the algebra generated by the Hasse-Schmidt derivations on A lifts to a subalgebra H of $D(R_m)$ that equals $D(R_m)$. Since $H \subseteq HS(R_m) \subseteq D(R_m)$, $HS(R_m) = D(R_m)$ for all maximal ideals m of R . Using Corollary 3.2.7 and the same argument, each Hasse-Schmidt derivation on R extends to a Hasse-Schmidt derivation on R_m and

$$HS(R) \otimes_R R_m = HS(R_m) = D(R_m).$$

Now consider the R -module $HS^n(R/k) = D^n(R/k) \cap HS(R/k)$. Because $P_{R/k}^n$ is finitely presented, $D^n(R/k) = \text{Hom}_R(P_{R/k}^n, R)$ is a finitely generated R -module. Now $\frac{D^n(R/k)}{HS^n(R/k)}$ is locally zero:

$$\frac{D^n(R/k)}{HS^n(R/k)} \otimes_R R_m \cong \frac{D^n(R/k) \otimes_R R_m}{HS^n(R/k) \otimes_R R_m} = \frac{D^n(R_m/k)}{HS^n(R_m/k)} = 0.$$

So $D^n(R/k) = HS^n(R/k)$. Because this holds for all n and $D(R/k) = \varinjlim D^n(R/k)$, the ring of differential operators $D(R/k)$ equals its subalgebra $HS(R/k)$. \square

Corollary 3.3.2. *If R is a smooth k -algebra of finite type, where k is a field of characteristic zero, then $D(R/k)$ is generated by derivations.*

Proof. In characteristic zero, $HS(R/k) = \text{der}(R/k)$. The result now follows from Theorem 3.3.1. \square

Remark 3.3.3. From the proof of the theorem, we see that the theorem and its corollary also hold for k -algebras R that are the localization of a smooth k -algebra of finite type.

Now we can show that if R is a smooth algebra over a field, then R is a finite product of D -simple domains.

Corollary 3.3.4. *If R is a smooth algebra over a field k , then R is a finite product of D -simple domains.*

Proof. Since R is smooth over a field, R is regular (Lemma 3.1.8). Then R is a product of domains. If a product of rings is smooth, then each factor is smooth, so R is a product of smooth domains. Now without loss of generality, we may suppose that R is a smooth domain.

Now we reduce to the local case. Suppose that R_m is $D(R_m/k)$ -simple for all maximal ideals m of R . If $I \subset R$ is a $D(R/k)$ -stable proper ideal, then there is a maximal ideal m containing I . Because R_m is a domain, IR_m is a nonzero $D(R_m/k)$ -stable proper ideal, a contradiction.

So suppose that (R, m) is a localization of a smooth k -algebra. Let x_1, \dots, x_d be a regular system of parameters for m . Suppose that I is an ideal of R and that $r \in I$ is nonzero. Then there is some integer N such that $r \in m^N \setminus m^{N+1}$. Using multi-index notation, write

$$\left[r - \sum_{|\mathbf{s}|=N} c_{\mathbf{s}} x^{\mathbf{s}} \right] \in m^{N+1},$$

where the $c_{\mathbf{s}}$ are units (or zero) and one of the $c_{\mathbf{s}}$ is nonzero, say $c_{\mathbf{e}}$. By Theorem 3.3.1, there is a differential operator D , such that D has order $\leq N$ and

$$D(x^{\mathbf{s}}) = \binom{\mathbf{s}}{\mathbf{e}} x^{\mathbf{s}-\mathbf{e}}.$$

The operator D is the extension of the operator

$$\partial^{\mathbf{e}} = \frac{1}{e_1!} \frac{\partial^{e_1}}{\partial x_1^{e_1}} \cdots \frac{1}{e_n!} \frac{\partial^{e_n}}{\partial x_n^{e_n}}$$

on the intermediate algebra $A = k[\mathbf{x}] = k[x_1, \dots, x_n]$ appearing in Theorem 3.3.1. Let $t = c_{\mathbf{e}}^{-1}r \in I$. Then $D(t) = 1 + D(m^{N+1})$. Since D has order $\leq N$, $D(m^{N+1}) \subseteq m$ and so $D(t)$ is a unit in the local ring R . Thus $I = R$. \square

Chapter 4

Stable Ideals

The D-module structure of the ring R is a very powerful tool in understanding the algebraic structure of both the ring R and its ring of differential operators $D(R)$. For instance, Smith used the fact that a variant of the test ideal of R is D-stable to characterize certain strongly F-regular rings in terms of D-simplicity [47] (these terms are defined in the last section of this chapter).

A detailed study of the D-module structure of R also has important implications for Nakai's conjecture. The Nakai hypothesis, $D(R) = HS(R)$, puts significant constraints on the D-module structure of R and this leads to restrictions on R itself. This theme is explored in chapter 6, where it is the main tool in showing that Nakai's conjecture holds for varieties whose normalization is smooth.

To begin, we treat a number of elementary results concerning the structure of R as a module over the rings $der(R)$, $HS(R)$, and $D(R)$. The next section deals with the relationship between stability and primary decomposition. In Theorem 4.2.4, we show that the associated primes of a HS -

stable ideal are HS -stable and that each HS -stable ideal admits a primary decomposition into HS -stable ideals. This extends an important result of Seidenberg concerning der -stable ideals. In the third section, we prove that Hasse-Schmidt derivations on a ring R extend to its integral closure R' and that the conductor ideal of R' into R is $HS(R)$ -stable. These results will reappear in our treatment of Nakai's conjecture. The proofs use the notion of quasi-integral closure, which is briefly reviewed. The last section surveys Smith's characterization of strongly F -regular rings in terms of their D -module structure.

4.1 Elementary results

This section treats a few elementary results about stable ideals. Since the Hasse-Schmidt algebra $HS(R)$ is an analogue of the derivation algebra $der(R)$, we expect that operations preserving $der(R)$ -stability will also preserve $HS(R)$ -stability. In general, the persistence of $D(R)$ -stability is a much more delicate issue.

Theorem 4.1.1. *Sums and intersections of HS -stable ideals are HS -stable. Similarly, sums and intersections of D -stable ideals are D -stable and sums and intersections of der -stable ideals are der -stable.*

Proof. These statements hold for modules over any ring. Stable ideals are just submodules of R over the appropriate ring ($D(R/k)$, $HS(R/k)$ or $der(R/k)$). □

Theorem 4.1.2. *Products of HS -stable ideals are HS -stable. Similarly, products of der -stable ideals are der -stable.*

Proof. Suppose I and J are $HS(R/k)$ -stable ideals and $\Delta = \{\delta_t\}_{t=0}^\infty$ is a Hasse-Schmidt derivation on R . Let $i \in I$ and $j \in J$. Then

$$\delta_n(ij) = \sum_{t=0}^n \delta_t(i)\delta_{n-t}(j) \in IJ. \quad (*)$$

So products of $HS(R/k)$ -stable ideals are $HS(R/k)$ -stable. The statement about $der(R/k)$ -stable ideals follows from $(*)$ with $n = 1$. \square

Both products and powers of D-stable ideals can fail to be D-stable.

Example 4.1.3. The ideal $P = xR \subseteq R = \frac{k[x,y]}{(xy)}$ is $D(R/k)$ -stable. This follows from Theorem 4.2.1 since xR is a minimal primary component of the D-stable ideal $(0) = xyR$. Now we claim that the operator $\gamma = x\frac{1}{2!}\frac{\partial^2}{\partial x^2}$ induces a differential operator on R . It is enough to check that γ stabilizes the monomial ideal (xy) . If a and b be non-negative integers, then

$$\gamma(x^{1+a}y^{1+b}) = \binom{1+a}{2}x^ay^{1+b} \in (xy),$$

since either xy divides x^ay^{1+b} or $\binom{1+a}{2} = 0$. Now it is easy to see that the ideal P^2 is not D-stable because

$$\gamma * x^2 = x \notin P^2.$$

4.2 Stability and primary decomposition

Theorem 4.2.1. *The minimal primary components of HS-ideals are HS-ideals. Similarly for $D(R)$ and $der(R)$.*

Proof. Let $A(R/k)$ be any of the R -algebras $D(R/k)$, $HS(R/k)$ or $der(R/k)$. Let $I \subseteq R$ be an $A(R/k)$ -stable ideal. The primary component of I corresponding to the minimal prime P of I is $IR_P \cap R$. Note that IR_P is stable

under the action of $A(R/k) \otimes_R R_P$. Take $\theta \in A(R/k)$ and $b \in IR_P \cap R$. Then $\theta * b \in R$. The operator $\theta \otimes 1$ is in $A(R/k) \otimes_R R_P$ and

$$(\theta \otimes 1) * (b \otimes 1) = (\theta * b) \otimes 1 \in I \otimes_R R_P = IR_P.$$

So $\theta * b \in IR_P \cap R$ and $IR_P \cap R$ is $A(R/k)$ -stable. □

The aim of this section is to show that *every* associated prime of a HS -ideal is HS -stable. Not every primary component of a HS -ideal need be HS -stable (as we will show in Example 4.2.9), but every HS -ideal admits a primary decomposition into HS -stable ideals. In order to establish these results, we will need some elementary facts about the extension of (primary) ideals of R to $R[[t]]$.

From now on, unless explicitly stated otherwise, we will always assume that the commutative rings we discuss are Noetherian. This guarantees that primary decompositions exist. Since any ideal I of R is finitely generated, the extension of I to $R[[t]]$ admits a simple description:

$$I[[t]] := I(R[[t]]) = \left\{ \sum_{n=0}^{\infty} a_n t^n : a_n \in I \right\}.$$

From this, it easily follows that

$$(I :_R J)[[t]] = (I[[t]] :_{R[[t]]} J[[t]]),$$

and

$$(I \cap J)[[t]] = I[[t]] \cap J[[t]].$$

Now let $\Delta = \{\delta_i\} \subset HS(R/k)$ be a Hasse-Schmidt derivation and let I be an ideal of R stable under the action of each component δ_i of Δ . For $a \in I$,

$$e^{t\Delta}(a) = \left(\sum_{i=0}^{\infty} t^i \delta_i \right) * a = \sum_{i=0}^{\infty} t^i (\delta_i * a) \in I[[t]]. \quad (*)$$

Because $e^{t\Delta}$ commutes with multiplication by t , $(*)$ implies that $e^{t\Delta}(I[[t]]) \subseteq I[[t]]$. Conversely, suppose that $e^{t\Delta}(I[[t]]) \subseteq I[[t]]$. Then $(*)$ shows that for $a \in I$, $\delta_i * a \in I$. It follows that $\delta_i(I) \subseteq I$ for all components δ_i . We summarize these observations in the following lemma.

Lemma 4.2.2. *Let $\Delta = \{\delta_i\} \subset HS(R/k)$ be a Hasse-Schmidt derivation. An ideal $I \subseteq R$ is stable under the action of each δ_i if and only if $I[[t]]$ is $e^{t\Delta}$ stable.*

For later use, we note that primality extends from R to $R[[t]]$.

Lemma 4.2.3. *If I is P -primary in R , then $I[[t]]$ is $P[[t]]$ -primary in $R[[t]]$.*

Proof. It is easy to check that the radical of $I[[t]]$ is $P[[t]]$ and that $P[[t]]$ is a prime ideal of $R[[t]]$. It remains to show that $P[[t]]$ is the only associated prime of $I[[t]]$. Because P is an associated prime of I , there exists $x \in R$, such that $P = (I :_R x)$. Then $P[[t]] = (I[[t]] :_{R[[t]]} x)$, so $P[[t]]$ is an associated prime of $I[[t]]$. Suppose that $Q \subset R[[t]]$ is another associated prime of $I[[t]]$. Because the radical of $I[[t]]$ is $P[[t]]$, $P[[t]] \subseteq Q$. Aiming for a contradiction, suppose that $P[[t]] \neq Q$ and take an element $a = a_i t^i + a_{i+1} t^{i+1} + \dots$ of $Q \setminus P[[t]]$ with $a_i \notin P$. Because Q is associated to $I[[t]]$, there exists $y = y_j t^j + y_{j+1} t^{j+1} + \dots$ in $R[[t]] \setminus I[[t]]$ such that

$$Q = (I[[t]] :_{R[[t]]} y).$$

Without loss of generality, $y_j \notin I$. The equation $ay \in I[[t]]$ forces $a_i y_j \in I$ and this contradicts our assumptions on a_i, y_j and the hypothesis that I is P -primary. It follows that $P[[t]]$ is the only associated prime of $I[[t]]$ and so $I[[t]]$ is $P[[t]]$ -primary. \square

Now we are ready to prove the results about HS -ideals alluded to earlier. The following theorem generalizes results of Seidenberg [44] about der -stable ideals. A partial proof of parts of the theorem also appeared in [5].

Theorem 4.2.4. *The associated primes of $HS(R)$ -stable ideals are $HS(R)$ -stable and every $HS(R)$ -stable ideal admits a primary decomposition with $HS(R)$ -stable components.*

Proof. The proof follows Seidenberg's original line of argument. Let I be a $HS(R)$ -stable ideal and let Δ be a Hasse-Schmidt derivation. Write

$$I = q_1 \cap \cdots \cap q_s,$$

where each q_i is p_i -primary. Then

$$I[[t]] = q_1[[t]] \cap \cdots \cap q_s[[t]],$$

and each $q_i[[t]]$ is $p_i[[t]]$ -primary by Lemma 4.2.3. Applying the automorphism $e^{t\Delta}$ we have $e^{t\Delta}I[[t]] = e^{t\Delta}q_1[[t]] \cap \cdots \cap e^{t\Delta}q_s[[t]]$. Also, $e^{t\Delta}q_i[[t]]$ is $e^{t\Delta}p_i[[t]]$ -primary. Because I is $HS(R)$ -stable, Lemma 4.2.2 forces $I[[t]]$ to be stable under the action of $e^{t\Delta}$ and so $e^{t\Delta}$ permutes the $p_i[[t]]$. Say $e^{t\Delta}p_i[[t]] = p_j[[t]]$. Then for $a \in p_i$, $e^{t\Delta}(a) = a + \delta_1(a)t + \cdots \in p_j[[t]]$, so $a \in p_j$. It follows that $p_i \subseteq p_j$. If Δ' is the inverse to Δ (in the group of Hasse-Schmidt derivations), then by Lemma 2.2.7, $e^{t\Delta'}$ is the inverse automorphism to $e^{t\Delta}$. Now $p_i[[t]] = e^{t\Delta'}p_j[[t]]$, so $p_j \subseteq p_i$. Therefore, $p_i = p_j$ and each prime p_i is stable under the action of the components of Δ . As Δ was an arbitrary Hasse-Schmidt derivation, each p_i is $HS(R)$ -stable.

To show that the $HS(R)$ -stable ideal I admits a primary decomposition with $HS(R)$ -stable components, suppose that $p_i^{r_i} \subseteq q_i$ and fix the integers r_i . Then

$$I[[t]] = q_1[[t]] \cap \cdots \cap q_s[[t]],$$

and $(p_i[[t]])^{r_i} = (p_i)^{r_i}[[t]] \subseteq q_i[[t]]$. Consider all representations $I[[t]] = Q_1 \cap \cdots \cap Q_s$ with Q_i a $p_i[[t]]$ -primary ideal and $(p_i[[t]])^{r_i} \subseteq Q_i$. Write the intersection of all these representations as

$$I[[t]] = \bar{q}_1 \cap \cdots \cap \bar{q}_s.$$

Then \bar{q}_i is $p_i[[t]]$ -primary, $(p_i[[t]])^{r_i} \subseteq \bar{q}_i$, and if $I[[t]] = q_1^* \cap \cdots \cap q_s^*$ is a representation of $I[[t]]$ of the kind mentioned above, then $\bar{q}_i \subseteq q_i^*$. Applying the automorphism $e^{t\Delta}$ we see that $\bar{q}_i \subseteq e^{t\Delta}(\bar{q}_i)$. Similarly, $\bar{q}_i \subseteq e^{t\Delta'}\bar{q}_i$. So $\bar{q}_i = e^{t\Delta}(\bar{q}_i)$ and \bar{q}_i is stable under the action of $e^{t\Delta}$.

Now we show that $q_i = q'_i[[t]]$ for a p_i -primary ideal q'_i . Let $q'_i = q_i \cap R$. Then q'_i is p_i -primary, $p_i^{r_i} \subseteq q'_i$ and $q'_i[[t]] \subseteq \bar{q}_i$. From $I[[t]] \cap R = I$, we see that $I[[t]] = q'_1[[t]] \cap \cdots \cap q'_s[[t]]$, and from the minimal property of the \bar{q}_i we see that $\bar{q}_i \subseteq q'_i[[t]]$. Hence $\bar{q}_i = q'_i[[t]]$. Now $q'_i[[t]]$ is stable under the action of $e^{t\Delta}$, so q'_i is $HS(R)$ -stable. \square

Theorem 4.2.4 leads to an easy proof that the radical of a HS -ideal is HS -stable.

Corollary 4.2.5. *The radical of a HS -stable ideal is HS -stable.*

Proof. The radical of an ideal is the intersection of its minimal primes. By Theorem 4.2.4, the minimal primes of a HS -stable ideal are themselves HS -stable and by Theorem 4.1.1 their intersection is also HS -stable. \square

Since the Hasse-Schmidt algebra agrees with the derivation algebra in a ring of characteristic zero, we immediately recover Seidenberg's results on derivations.

Corollary 4.2.6 (Seidenberg [44]). *If R is a ring of characteristic zero, then the associated primes of $der(R)$ -stable ideals are $der(R)$ -stable and*

every $\text{der}(R)$ -stable ideal admits a primary decomposition with $\text{der}(R)$ -stable components. Moreover, the radical of a der -stable ideal is der -stable.

Example 4.2.7. In prime characteristic, the minimal primes and the radical of a $\text{der}(R)$ -stable ideal need not be $\text{der}(R)$ -stable. For example, let $R = \frac{\mathbb{F}_p[x]}{(x^p)}$. Then $\frac{d}{dx} \in \text{Der}(R)$ and $(0)R = x^p R$ is $\text{Der}(R)$ -stable but $\sqrt{(0)R} = xR$ is not $\text{Der}(R)$ -stable. This provides another example of a derivation $\frac{d}{dx}$ that is not the first component of a Hasse-Schmidt derivation (cf. Example 2.2.3).

The minimal primes and the radical of a D -stable ideal may each fail to be D -stable.

Example 4.2.8. We study a nonreduced scheme supported on a single point. Consider $R = \frac{k[x_1, \dots, x_N]}{(x_1^2, \dots, x_N^2)}$, where k is a field. Then we claim that $D(R/k) = \text{End}_k(R)$. Note that if θ is any k -endomorphism, then for any $r_0, \dots, r_{2N} \in (x_1, \dots, x_N)$, we can expand

$$[\dots [[\theta, r_0], r_1], \dots, r_{2N}]$$

into a sum of terms of the form $\pm r_{i_0} \cdots r_{i_t} \theta r_{i_{t+1}} \cdots r_{i_{2N}}$. These terms are zero since any power of any x_i is zero in R . So $\text{End}_k(R) = D(R/k)$. Since R is a finite dimensional k -vector space, $\text{End}_k(R)$ is a matrix ring and it is simple (see Lam [28, Theorem 3.3]). Thus, $D(R/k)$ is a simple ring.

Now we claim that R is a simple $D(R/k)$ -module. If $I \subset R$ is a proper D -stable ideal, then $\text{Ann}_{D(R)}(R/I)$ is a proper two-sided ideal of $D(R)$. Since $D(R)$ is a simple ring, $\text{Ann}_{D(R)}(R/I) = 0$. But $I(R/I) = 0$, so $I = 0$ and R is a simple $D(R/k)$ -module.

In this example (0) is a D -ideal, but its only associated prime $\sqrt{(0)} = (\bar{x}_1, \dots, \bar{x}_N)$ is not a D -ideal (as R is D -simple).

The next example shows that not every embedded primary component of a HS -stable ideal is HS -stable.

Example 4.2.9. Consider the ring $R = \frac{k[x,y]}{(x^2,xy)}$. Then (0) is a D -stable ideal with primary decomposition $(0) = (y)R \cap (x)R$. Here, (x) is a minimal component and (y) is an embedded component (with radical (x,y)). It is easy to check that $\theta = x\partial_x\partial_y - \partial_y \in D(R)$: $(x\partial_x\partial_y - \partial_y) * x^N = 0$; for $x^a y^b \in (x^2, xy)$, $(x\partial_x\partial_y - \partial_y) * x^a y^b = (ab - b)x^a y^{b-1}$ and $x^a y^{b-1}$ is zero unless $a = b = 1$, in which case, $ab - b = 0$. On the other hand, y is in an embedded primary component $q = (y)$ of (0) and $\theta * y = -1 \notin q$. So embedded primary components (and embedded associated primes) need not be D -stable. This example is independent of the characteristic of k .

Still considering the same ring, $R = \frac{k[x,y]}{(x^2,xy)}$, we treat HS -stability. Set $\delta_n = x^n \frac{1}{n!} \frac{\partial^n}{\partial y^n}$. Then we claim that $\Delta = \{\delta_t\}_{t=0}^\infty$ is a Hasse-Schmidt derivation on $k[x,y]$ that restricts to a Hasse-Schmidt derivation on R . First we check that $I = (x^2, xy)$ is stable under δ_n . It is enough to check that δ_n sends the generators of I into I . Now $\delta_n(x^2) = 0$ and

$$\delta_n(xy) = \begin{cases} xy & \text{if } n = 0, \\ x^2 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now we check that Δ defines a Hasse-Schmidt derivation on $k[x,y]$. Since each δ_t preserves total degree, it is sufficient to check that $\delta_n(\alpha\beta) =$

$\sum_{t=0}^n \delta_t(\alpha)\delta_{n-t}(\beta)$ for monomials α and β :

$$\begin{aligned}
\delta_n(x^a y^b \cdot x^c y^d) &= \delta_n(x^{a+c} y^{b+d}) \\
&= \binom{b+d}{n} x^{a+c+n} y^{b+d-n} \\
&= \left[\sum_{t=0}^n \binom{b}{t} \binom{d}{n-t} \right] x^{a+c+n} y^{b+d-n} \\
&= \sum_{t=0}^n \binom{b}{t} x^{a+t} y^{b-t} \binom{d}{n-t} x^{c+n-t} y^{d-n+t} \\
&= \sum_{t=0}^n \delta_t(x^a y^b) \delta_{n-t}(x^c y^d).
\end{aligned}$$

An equally easy proof that Δ induces a Hasse-Schmidt derivation on R results from the observation that $\delta_n = 0$ for $n \geq 2$.

The ideal $(0)R = (x^2, xy)R$ is $HS(R/k)$ -stable and admits a primary decomposition

$$(x^2, xy)R = (x) \cap (x^2, y),$$

where (x^2, y) is an embedded primary component. Now

$$\delta_1(y) = x \frac{\partial}{\partial y} * y = x \notin (x^2, y),$$

so not every embedded component of a HS -stable ideal is HS -stable. Also, since δ_1 is clearly a derivation, the embedded component (x^2, y) is not $der(R/k)$ -stable.

Theorem 4.2.4 guarantees that I admits a primary decomposition into HS -stable ideals. For instance,

$$I = xR \cap y^2R.$$

Here, xR is a minimal primary component of I and hence xR is HS -stable. The ideal $y^2R = (x^2, xy, y^2)R = (x, y)^2R$ is clearly primary to the homogeneous maximal ideal $m = (x, y)$. Since m is an associated prime of I , m is HS -stable and by Lemma 4.1.2, $y^2R = m^2$ is HS -stable. The components of this primary decomposition are also der -stable. Because every power of m is HS -stable, every Hasse-Schmidt derivation on R must preserve degree.

The ideal quotient behaves well with respect to *HS*-stability and *der*-stability but not with respect to D-stability.

Theorem 4.2.10. *If I and J are HS-stable ideals, then $(I : J)$ is a HS-stable ideal. Similarly, if I and J are der-stable ideals, then $(I : J)$ is a der-stable ideal.*

Proof. Let $\Delta = \{\delta_t\}$ be a *HS*-derivation. Take $a \in (I : J)$ and $j \in J$. We prove the statement

$$\delta_n(a) \in (I : J)$$

by induction on n . The case $n = 0$ is obvious since δ_0 is the identity map. To establish the case $n = N$, we may assume $\delta_m(a) \in (I : J)$ for $m < N$. Apply δ_N to the product aj :

$$\begin{aligned} \delta_N(aj) &= \sum_{s=0}^N \delta_s(a)\delta_{N-s}(j) \\ &= \delta_N(a)j + \underbrace{\delta_{N-1}(a)\delta_1(j) + \cdots + \delta_1(a)\delta_{N-1}(j)}_{\in I} + a\delta_N(j). \end{aligned} \quad (*)$$

Since $aj \in I$ and I is *HS*-stable, $\delta_N(aj) \in I$. Then $(*)$ forces $\delta_N(a)j \in I$. As j was an arbitrary element of J , $\delta_N(a)J \subseteq I$ and $\delta_N(a) \in (I : J)$. So $(I : J)$ is *HS*-stable. The assertion about *der*-stable ideals has a similar proof. \square

The ideal quotient of two D-stable ideals need not be D-stable.

Example 4.2.11. Let $R = \frac{k[x,y]}{(x^2,xy)}$. The minimal component of (0) is (x) and so (x) is D-stable. However, $((0) : (x)) = (x, y)$ and we showed in Example 4.2.9 that (x, y) is not D-stable.

4.3 The conductor is $HS(R)$ -stable

Seidenberg showed that derivations on a characteristic zero Noetherian domain, R , extend to the integral closure, R' , of R in its field of quotients, K . In this section, we extend this result to reduced rings of arbitrary characteristic. More precisely, we show that components of Hasse-Schmidt derivations extend to the integral closure R' of R in the total ring of quotients $L = S^{-1}R$, where S is the multiplicative set of all nonzero-divisors of R and R is a reduced Noetherian algebra of finite type defined over a ring k . This is then used to prove that the conductor C of R' into R , is stable under the action of the components of the Hasse-Schmidt derivations of R .

4.3.1 Quasi-integral closure

We begin with some preliminary remarks on quasi-integral closure. An element $a \in L$ is said to be quasi-integral over R if there exists some nonzero-divisor $b \in R$ such that $ba^n \in R$ for $n = 0, 1, \dots$. Recall that a is integral over R if a satisfies a monic equation with coefficients in R . In Noetherian rings, the two concepts are equivalent.

Proposition 4.3.1. If R is Noetherian then $a \in L$ is quasi-integral over R if and only if a is integral over R .

Proof. Suppose that a is integral over R . Then there is some relation of the form

$$a^n + r_1 a^{n-1} + r_2 a^{n-2} + \dots + r_n = 0,$$

where each $r_i \in R$. Write $a = \frac{r}{s}$ and let $b = s^{n-1}$. Then $ba^k \in R$ for $k = 0, 1, \dots, n-1$. The relation above says that a^n (and higher powers

of a) can be written as an R -linear combination of $\{1, a, \dots, a^{n-1}\}$, so it follows that $ba^k \in R$ for all k .

Conversely, assume that a is quasi-integral over R . Then

$$(b, ba, ba^2, \dots)$$

is an ideal in the Noetherian ring R , and so it is finitely generated. Then there is some n for which

$$ba^n = \sum_{i=0}^{n-1} r_i ba^i,$$

where each $r_i \in R$. Since b is not a zero-divisor, this gives the equation:

$$a^n = \sum_{i=0}^{n-1} r_i a^i,$$

and so a is integral over R . □

We will use the notion of quasi-integral closure rather than the notion of integral closure. Since all our rings are assumed to be Noetherian, these are equivalent; however, the definition of quasi-integral closure is better behaved under the action of Hasse-Schmidt derivations.

4.3.2 Hasse-Schmidt derivations extend

Theorem 4.3.2. *Let R be a reduced Noetherian k -algebra and let R' be its integral closure in the total quotient ring L of R . If $\Delta = \{d_n\}$ is a Hasse-Schmidt derivation on R , then Δ extends to a Hasse-Schmidt derivation on R' .*

Proof. Each d_n extends to a map of the total quotient ring to itself. One can check this directly or just use Corollary 3.2.7. As in chapter 2, this enables

us to define the map $E = e^{t\Delta} : L[[t]] \rightarrow L[[t]]$. This is the unique $k[t]$ -linear ring homomorphism such that

$$E(c) = \sum_{i=0}^{\infty} d_i(c)t^i,$$

for each $c \in L$. Note that since Δ is a Hasse-Schmidt derivation on R , $E(R[[t]]) \subseteq R[[t]]$.

Let $a \in R'$. Then by Theorem 4.3.1 a is quasi-integral over R , so there exists a nonzero-divisor $b \in R$ such that $ba^n \in R$ for all n . Because E is a ring homomorphism, $E(b)E(a)^n = E(ba^n) \in R[[t]]$ for all n . Now,

$$bE(b)(E(a) - a)^n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \underbrace{[E(b)E(a)^k]}_{\in R[[t]]} \underbrace{[ba^{n-k}]}_{\in R[[t]]} \in R[[t]].$$

The coefficient of the lowest order term in t in this expression is $b^2(d_1(a))^n$ and the expression says that this is in R . Since b is a nonzero-divisor in R , so is b^2 . Thus, $d_1(a)$ is quasi-integral over R . Then $d_1(a)$ is integral over R : $d_1(a) \in R'$.

Now we finish the proof using induction. Suppose that $b^{2^i}d_i(a)^n \in R$ for all n and for $i < k$. Then

$$b^{2^k-1}E(b)(E(a) - \sum_{s=0}^{k-1} d_s(a))^n = \sum_{i_0+\dots+i_k=n} (-1)^{n-i_k} \eta(\mathbf{i}) [ba^{i_0}] [b^2d_1(a)^{i_1}] \cdots [b^{2^{k-1}}d_k(a)^{i_{k-1}}] [E(b)E(a)^{i_k}] \in R[[t]],$$

where $\eta(\mathbf{i}) = \eta(i_0, i_1, \dots, i_k) = \frac{n!}{i_0!i_1!\cdots i_k!}$. The coefficient of the lowest order term in t in this expression is $b^{2^k}(d_k(a))^n$ and the expression says that this is in R . This induction forces $d_i(a)$ to be quasi-integral (and hence integral) over R for all i . Thus, each component of Δ extends to R' and Δ extends to a Hasse-Schmidt derivation on R' . \square

The conductor of R' into R is defined to be

$$C = \{c \in R : cR' \subseteq R\}.$$

Note that C is an ideal of both R and its integral closure R' . We deduce that the conductor is $\text{HS}(R/k)$ -stable from the fact that Hasse-Schmidt derivations extend to the integral closure.

Corollary 4.3.3. *The conductor of a reduced Noetherian k -algebra R is $\text{HS}(R/k)$ -stable.*

Proof. Suppose that $c \in C$ and $s \in R'$. Let $\{d_i\}$ be a Hasse-Schmidt derivation on R . Then

$$d_i(cs) = cd_i(s) + d_1(c)d_{i-1}(s) + \cdots + d_i(c)s \in R.$$

Because $d_j(s)$ is in R' , induction on i shows that $d_i(c) \in C$. □

4.4 D-stability and strong F-regularity

In the next chapter, our results on D-stability will be applied to show that varieties with smooth normalization satisfy Nakai's conjecture. This section surveys another topic associated with D-stability: Smith's characterization of strongly F-regular rings [47].

When R is a ring of prime characteristic p , the Frobenius map, $r \mapsto r^p$, is a ring endomorphism. This implies that the image of the Frobenius map, R^p , is a subring of R . Furthermore, this process can be iterated: R is an algebra over the image of the e^{th} iterate of the Frobenius map, $R^{p^e} \subset R$.

Recall that the ring R is said to be *F-finite* if R is a finite module over R^p (F stands for Frobenius). The ring R is said to be *F-split* if the map

$R^p \hookrightarrow R$ splits as a map of R^p -modules. The ring R is *strongly F-regular* if for all c not in any minimal prime of R , there is some power p^e of the characteristic such that the R^{p^e} -linear map

$$\begin{aligned} R^{p^e} &\rightarrow R \\ 1 &\mapsto c \end{aligned}$$

splits as a map of R^{p^e} -modules. It is worth noting that by Theorem 2.1.11, the splitting $R \rightarrow R^{p^e}$, followed by the inclusion $R^{p^e} \hookrightarrow R$ is a differential operator on R .

Strong F-regularity is primarily important because of its connection with the theory of tight closure; however, the method of reduction to characteristic p (see Hochster's appendix in [21]), together with the notion of strong F-regularity, can be used to obtain information about singularities of characteristic zero varieties. If R is a finitely generated algebra over a field K of characteristic zero, then there is a finitely generated \mathbb{Z} -algebra A such that $R = R_A \otimes_A K$, where $R = \frac{K[\mathbf{x}]}{I}$ and $R_A = \frac{A[\mathbf{x}]}{I \cap A[\mathbf{x}]}$. Then $\text{Spec } R_A$ fibers over $\text{Spec } \mathbb{Z}$ via the natural map $\mathbb{Z} \rightarrow R_A$. R is said to be of *strongly F-regular type* if the fibers over an open dense set of closed points of $\text{Spec } \mathbb{Z}$ are strongly F-regular (the fibers have coordinate rings $R_A \otimes_{\mathbb{Z}} \mathbb{Z}_p$ of prime characteristic). Recent work of Smith, Hara and Watanabe identify certain singularities arising in birational geometry with Frobenius-type conditions (see Smith's survey of these results in [49]). For instance, log-terminal singularities are those singularities with \mathbb{Q} -Gorenstein coordinate ring of strongly F-regular type. A normal local ring R is \mathbb{Q} -Gorenstein if the canonical module ω_R defines a torsion element in the ideal class group.

Smith has shown that strongly F-regular rings R are characterized in terms of the structure of R as a module over its ring of differential operators

and its Frobenius power R^p . In [47] Smith first shows that an asymptotic variant of the test ideal, an invariant of R associated with tight closure, is a $D(R)$ -stable ideal when R is F -finite. This, together with some results on tight closure, establish the following characterization of strongly F -regular rings:

Theorem 4.4.1 (Smith [47]). *If R is an F -finite ring of prime characteristic, then R is strongly F -regular if and only if R is both $D(R)$ -simple and F -split.*

Smith and Van den Bergh [51] have used this result and these methods to study the simplicity of $D(R)$ where R is a ring of invariants.

Chapter 5

Differential Operators on Stanley-Reisner Rings

It is often very difficult to give an explicit description of the ring of differential operators on a variety; however, this is feasible for unions of coordinate subspaces of affine space. The coordinate rings of such varieties are called Stanley-Reisner rings. These rings are a fundamental algebraic object that arise in many contexts: they can be associated to graphs [45] and, more generally, to simplicial complexes [14]. The monomial grading on a Stanley-Reisner ring can be used to translate algebraic questions into combinatorial ones. As well, the grading on a Stanley-Reisner ring R induces a powerful grading on its ring of differential operators $D(R)$. This allows a complete characterization of the ring of differential operators on a Stanley-Reisner ring of arbitrary characteristic, generalizing work of Brumatti and Simis [8] concerning derivations on Stanley-Reisner rings. The characterization can be phrased in terms of local cohomology and the generators of the R -module $D(R)$ can be described in terms of the simplicial complex associated to R .

The last section treats the D-module structure of Stanley-Reisner rings. We completely characterize the D-module structure of a Stanley-Reisner ring R defined over a field in terms of the minimal primes of R .

5.1 Preliminaries

We begin with a few definitions and preliminary results. A monomial ideal J is an ideal of a polynomial ring $k[\mathbf{x}] = k[x_1, \dots, x_N]$ (where k is a commutative domain) that is generated by monomials in x_1, \dots, x_N . If J is such an ideal, we call the quotient ring $\frac{k[\mathbf{x}]}{J}$ a monomial ring. As we have already remarked, reduced rings of this type are called Stanley-Reisner rings. If $\pi : k[\mathbf{x}] \rightarrow \frac{k[\mathbf{x}]}{J}$ is the natural projection, then we consistently abuse notation by writing y for $\pi(y)$. For instance, we say that a monomial term is the product of a nonzero element of k with a monomial in $k[\mathbf{x}]$.

Monomial ideals are prime if and only if they are generated by some subset of the variables. Monomial ideals are radical if and only if they are generated by squarefree monomials. Obtaining the minimal primes in a monomial ring is facilitated by the following well-known result.

Proposition 5.1.1. The minimal primes over a monomial ideal $I = (\{x^\mu\})$ are all of the form $p = (x_{i_1}, \dots, x_{i_t})$ where

- (1) Every minimal generator x^μ of I is divisible by some x_{i_j}
- (2) $\forall x_{i_j} \exists$ a minimal generator x^μ such that $x_{i_j} | x^\mu$ and no other x_{i_k} divides x^μ .

Note that condition (2) is equivalent to saying that the set $\{x_{i_1}, \dots, x_{i_t}\}$ is minimal with respect to condition (1).

If $R = \bigoplus R_i$ is a graded k -algebra, then a differential operator $\theta \in D(R/k)$ is said to have degree d if $\theta(R_i) \subseteq R_{i+d}$ for all i . The

polynomial ring $k[x_1, \dots, x_N]$ is graded by \mathbb{N}^N and the Weyl algebra $k[x_1, \dots, x_N, \dots, \frac{1}{n!} \frac{\partial^n}{\partial x_i^n}, \dots]$ is graded by \mathbb{Z}^N . We write ∂_i^n for the operator $\frac{1}{n!} \frac{\partial^n}{\partial x_i^n}$. For $\mathbf{a} \in \mathbb{N}^N$, the symbol $\partial^{\mathbf{a}}$ is to be interpreted as the composition of these operators: $\partial^{\mathbf{a}} = \partial_1^{a_1} \dots \partial_N^{a_N}$.

5.2 Description of $D(R)$

Set $R = \frac{k[x_1, \dots, x_N]}{J}$, where J is a monomial ideal of $k[x_1, \dots, x_N] = k[\mathbf{x}]$. To determine $D(R/k)$ it suffices to determine the idealizer of J . The next result shows that when R is reduced (a Stanley-Reisner ring), it suffices to determine which terms $x^{\mathbf{b}} \partial^{\mathbf{a}}$ are in the idealizer of J . This is the technical heart of the description of the ring of differential operators on a Stanley-Reisner ring. Later, we will give another derivation of the description using the theory of D-stability developed in chapter 4. The computations in the proof of the lemma are interesting in their own right; they depend on the different types of behavior of binomial coefficients in characteristics zero and p .

Theorem 5.2.1. *Let J be a radical monomial ideal of the ring $k[x_1, \dots, x_N]$, where k is a commutative domain. Then an element $\theta = \sum_{\mathbf{a}, \mathbf{b}} k_{\mathbf{a}, \mathbf{b}} x^{\mathbf{b}} \partial^{\mathbf{a}}$ ($k_{\mathbf{a}, \mathbf{b}} \in k$) is in the idealizer $\mathbb{I}(J)$ of J if and only if each term of θ is in $\mathbb{I}(J)$.*

Proof. Let $\theta = \sum_{\mathbf{v}} \theta_{\mathbf{v}}$ be an element of the Weyl algebra, where $\theta_{\mathbf{v}}$ is a differential operator of degree $\mathbf{v} \in \mathbb{Z}^N$. Since J is a homogeneous ideal in the \mathbb{N}^N -graded ring $k[x_1, \dots, x_N]$, $\theta \in \mathbb{I}(J)$ if and only if each graded piece of θ is in $\mathbb{I}(J)$; that is, if and only if each $\theta_{\mathbf{v}} \in \mathbb{I}(J)$. So, without loss of generality, we can assume that all the terms in $\theta = \sum_{\mathbf{a}, \mathbf{b}} k_{\mathbf{a}, \mathbf{b}} x^{\mathbf{b}} \partial^{\mathbf{a}}$ are of

degree \mathbf{v} . Then $\theta = \sum_{\mathbf{a} \in S \subset \mathbb{N}^N} k_{\mathbf{a}} x^{\mathbf{a}+\mathbf{v}} \partial^{\mathbf{a}}$. Aiming for a contradiction, we can assume that no term of θ is in $\mathbb{I}(J)$.

Pick one term $k_{\mathbf{c}} x^{\mathbf{c}+\mathbf{v}} \partial^{\mathbf{c}}$ of $\theta = \sum_{\mathbf{a} \in S} k_{\mathbf{a}} x^{\mathbf{a}+\mathbf{v}} \partial^{\mathbf{a}}$. As this term is not in $\mathbb{I}(J)$, there is a monomial $x^{\eta} \in J$ such that $k_{\mathbf{c}} x^{\mathbf{c}+\mathbf{v}} \partial^{\mathbf{c}} * x^{\eta} = k_{\mathbf{c}} \binom{\eta_1}{c_1} \cdots \binom{\eta_N}{c_N} x^{\eta+\mathbf{v}} \notin J$. As J is radical, J is the intersection of its minimal primes. So $k_{\mathbf{c}} x^{\mathbf{c}+\mathbf{v}} \partial^{\mathbf{c}} * x^{\eta}$ is not in some minimal prime P of J . In particular, $x^{\eta+\mathbf{v}} \notin P$. Relabel if necessary to get $P = (x_1, \dots, x_d)$.

Let $\mathbf{m} = (0, \dots, 0, m_{d+1}, \dots, m_N) \in \mathbb{N}^N$ and note that $x^{\mathbf{m}} \notin (x_1, \dots, x_d) = P$. As $x^{\eta} \in J$, $x^{\eta+\mathbf{m}} \in J$. Since $\theta \in \mathbb{I}(J)$,

$$\theta * x^{\eta+\mathbf{m}} = \sum_{\mathbf{a}} k_{\mathbf{a}} \binom{\eta_1}{a_1} \cdots \binom{\eta_d}{a_d} \binom{\eta_{d+1} + m_{d+1}}{a_{d+1}} \cdots \binom{\eta_N + m_N}{a_N} x^{\eta+\mathbf{v}+\mathbf{m}} \in J \subset P.$$

Together with $x^{\eta+\mathbf{v}+\mathbf{m}} = x^{\eta+\mathbf{v}} x^{\mathbf{m}} \notin P$, this implies that

$$\sum_{\mathbf{a}} k_{\mathbf{a}} \binom{\eta_1}{a_1} \cdots \binom{\eta_d}{a_d} \binom{\eta_{d+1} + m_{d+1}}{a_{d+1}} \cdots \binom{\eta_N + m_N}{a_N} = 0, \quad (5.1)$$

for all $m_i \geq 0$ ($i \geq d+1$).

We make some remarks that simplify the sum (5.1). To start, observe that for each term $k_{\mathbf{a}} x^{\mathbf{a}+\mathbf{v}} \partial^{\mathbf{a}}$ in $\theta = \sum_{\mathbf{a} \in S} k_{\mathbf{a}} x^{\mathbf{a}+\mathbf{v}} \partial^{\mathbf{a}}$, we have $a_i \geq \eta_i$ for $i \leq d$. To see this first note that $a_i + v_i \geq 0$ for all i and so $a_i \geq -v_i$ for all i . Then, since $k_{\mathbf{c}} x^{\mathbf{c}+\mathbf{v}} \partial^{\mathbf{c}} * x^{\eta} \notin P = (x_1, \dots, x_d)$, we have $x^{\eta+\mathbf{v}} \notin (x_1, \dots, x_d)$ and so $\eta_i + v_i = 0$ for $i \leq d$. Together with our previous result, this gives $a_i \geq -v_i = \eta_i$ for $i \leq d$.

Also, $c_i = \eta_i$ for each $i \leq d$ since if $c_i > \eta_i$ then $\binom{\eta_i}{c_i} = 0$ and so $k_{\mathbf{c}} x^{\mathbf{c}+\mathbf{v}} \partial^{\mathbf{c}} * x^{\eta} = 0 \in P$ (a contradiction).

Now, many of the terms in sum 5.1 are zero, resulting in:

$$\sum_{\mathbf{a} \in S: a_1=\eta_1, \dots, a_d=\eta_d} k_{\mathbf{a}} \binom{\eta_{d+1} + m_{d+1}}{a_{d+1}} \cdots \binom{\eta_N + m_N}{a_N} = 0, \quad (\dagger)$$

for all $m_i \geq 0$ ($i \geq d+1$). Note that this sum is nonempty since $k_{\mathbf{c}}$ appears.

Now our proof diverges into two cases, depending on the characteristic of k .

If the characteristic of k is 0: Let $F = \text{frac}(k)$ be the fraction field of k . Equation (\dagger) continues to hold in F . The key point is that equation (\dagger) holds for all $(0, \dots, 0, m_{d+1}, \dots, m_N) \in \mathbb{N}^N$, so we can think of (\dagger) as a polynomial in the indeterminates m_{d+1}, \dots, m_N with coefficients in F . Here we are using the fact, over a characteristic zero field, $\binom{x}{d}$ is a polynomial in the variable x of degree d . Since the polynomial (\dagger) is zero on all natural numbers, it is the zero polynomial.

The polynomial $\binom{\eta_{d+1}+m_{d+1}}{a_{d+1}} \dots \binom{\eta_N+m_N}{a_N}$ is an element in the multi-graded ring $F[m_{d+1}, \dots, m_N]$ which has a unique nonzero term of multi-degree (a_{d+1}, \dots, a_N) (its leading term). It follows that the polynomials $\binom{\eta_{d+1}+m_{d+1}}{a_{d+1}} \dots \binom{\eta_N+m_N}{a_N}$ are linearly independent over F . Interpreting (\dagger) as an equality of polynomials, linear independence implies $k_{\mathbf{c}} = 0$. This contradicts $k_{\mathbf{c}}x^{\mathbf{c}+\mathbf{v}}\partial^{\mathbf{c}} \notin \mathbb{I}(J)$.

If the characteristic of k is $p > 0$: Define a relation on the indexing set S of the sum $\theta = \sum_{\mathbf{a} \in S} k_{\mathbf{a}}x^{\mathbf{a}+\mathbf{v}}\partial^{\mathbf{a}}$ as follows. We say $\mathbf{a} = (a_1, \dots, a_N) \preceq \mathbf{b} = (b_1, \dots, b_N)$ if $(a_{d+1}, \dots, a_N) \leq (b_{d+1}, \dots, b_N)$ in the lex order induced by the following ordering on the integers: $n = \sum_{i=0}^t n_i p^i \leq m = \sum_{i=0}^t m_i p^i$ ($0 \leq n_i, m_i < p$ for all i) if $(n_0, n_1, \dots, n_t) \leq (m_0, m_1, \dots, m_t)$ in the lex ordering induced by the usual ordering on the integers $0, 1, \dots, p-1$.

Let T be the set of indices \mathbf{a} which agree with η in their first d components and such that $k_{\mathbf{a}}x^{\mathbf{a}+\mathbf{v}}\partial^{\mathbf{a}} * x^{\eta} \notin J$ (that is, $k_{\mathbf{a}}\binom{\eta_1}{a_1} \dots \binom{\eta_N}{a_N} \neq 0$). This is not an empty set since $\mathbf{c} \in T$. Note that \preceq restricts to a total ordering on T . Let \mathbf{c}' be minimal in T .

Note that $k_{\mathbf{c}'}$ appears in the sum (\dagger) . We will show that for a particular

choice of the m_i , the sum (†) has only one summand, $k_{\mathbf{c}'}$, and so $k_{\mathbf{c}'} = 0$.

This will be a contradiction to $k_{\mathbf{c}'} x^{\mathbf{c}'+\mathbf{v}} \partial^{\mathbf{c}'} * x^\eta \notin J$.

If $\mathbf{a} \in S \setminus T$ but $a_1 = \eta_1, \dots, a_d = \eta_d$, then

$$k_{\mathbf{a}} x^{\mathbf{a}+\mathbf{v}} \partial^{\mathbf{a}} * x^\eta = k_{\mathbf{a}} \binom{\eta_{d+1} + m_{d+1}}{a_{d+1}} \dots \binom{\eta_N + m_N}{a_N} x^{\eta+\mathbf{v}} \in J.$$

Together with $x^{\eta+\mathbf{v}} \notin J$, this implies $k_{\mathbf{a}} \binom{\eta_{d+1} + m_{d+1}}{a_{d+1}} \dots \binom{\eta_N + m_N}{a_N} = 0$. This implies that the indexing set in equation (†) simplifies further:

$$\sum_{\mathbf{a} \in T} k_{\mathbf{a}} \binom{\eta_{d+1} + m_{d+1}}{a_{d+1}} \dots \binom{\eta_N + m_N}{a_N} = 0, \quad (5.2)$$

for all $m_i \geq 0$ ($i \geq d+1$).

Recall that \mathbf{c}' is minimal in T under \preceq . Then for large e , $m_i = c'_i - \eta_i + p^e$ is nonnegative and equation (5.2) holds, so that

$$\sum_{\mathbf{a} \in T} k_{\mathbf{a}} \binom{c'_{d+1} + p^e}{a_{d+1}} \dots \binom{c'_N + p^e}{a_N} = 0. \quad (5.3)$$

Writing base p expansions of c'_i and a_i , we have $c'_i = \sum_{j=0}^t c'_{i,j} p^j$ and $a_i = \sum_{j=0}^t a_{i,j} p^j$ ($0 \leq a_{i,j}, c'_{i,j} < p$). So equation 5.3 becomes

$$\sum_{\mathbf{a} \in T} k_{\mathbf{a}} \left(\prod_{j=0}^t \binom{c'_{d+1,j}}{a_{d+1,j}} \right) \dots \left(\prod_{j=0}^t \binom{c'_{N,j}}{a_{N,j}} \right) = 0. \quad (5.4)$$

Here we have used the expansion of binomial coefficients in characteristic p (Lemma 2.1.10) and the fact that $\binom{1}{0} = 1$.

The sum in equation (5.4) has only one nonzero term, indexed by \mathbf{c}' . This follows from the minimality of \mathbf{c}' . To see this, observe that for each $\mathbf{a} \in T$, $\mathbf{a} \succeq \mathbf{c}'$ so if $a_{i,j} \leq c'_{i,j}$ for all $i = d+1, \dots, N$ and $j = 0, \dots, t$ then $\mathbf{a} = \mathbf{c}'$. The only instance in which the product $\left(\prod_{j=0}^t \binom{c'_{d+1,j}}{a_{d+1,j}} \right) \dots \left(\prod_{j=0}^t \binom{c'_{N,j}}{a_{N,j}} \right)$ does not vanish is when $a_{i,j} = c'_{i,j}$ for all $i = d+1, \dots, N$ and all $j = 0, \dots, t$; that is, when $a_i = c'_i$ for all $i \geq d+1$. This condition is only satisfied by

\mathfrak{c}' . It follows that $k_{\mathfrak{c}'}$ is the only term in sum (5.4). This gives $k_{\mathfrak{c}'} = 0$, a contradiction. \square

Example 5.2.2. Theorem 5.2.1 is not true when the ring R is not reduced. For example, let $R = \frac{k[x,y]}{(x^p, x^{p-1}y)}$, where p is the characteristic of k if k has positive characteristic and is an integer ≥ 2 if k has characteristic 0. Then $((p-1)\partial_2 - x\partial_1\partial_2) \in D(R)$. To see this, first note that $((p-1)\partial_2 - x\partial_1\partial_2) * x^a y^b = 0$. Now let $x^a y^b \in J$ with $a \geq p-1$ and $b \geq 1$. Then

$$((p-1)\partial_2 - x\partial_1\partial_2) * x^a y^b = ((p-1)b - ab)x^a y^{b-1}$$

and $x^a y^{b-1} \in J$ unless $a = p-1$ and $b = 1$, in which case, $(p-1)b - ab = 0$. So $((p-1)\partial_2 - x\partial_1\partial_2) \in D(R)$. However, $(p-1)\partial_2$ and $-x\partial_1\partial_2$ are not in $D(R)$ since they send $x^{p-1}y \in J$ to $\pm(p-1)x^{p-1} \notin J$.

We are now in a position to determine the ring of differential operators on a Stanley-Reisner ring, $R = \frac{k[\mathbf{x}]}{J}$.

Theorem 5.2.3. *Given a Stanley-Reisner ring $R = \frac{k[\mathbf{x}]}{J}$, an element of the Weyl algebra $x^{\mathbf{b}}\partial^{\mathbf{a}}$ is in $D(R/k)$ if and only if for each minimal prime P of R , we have either $x^{\mathbf{b}} \in P$ or $x^{\mathbf{a}} \notin P$. That is,*

$$x^{\mathbf{b}} \in \bigcap_{x^{\mathbf{a}} \in P_i} P_i,$$

where the P_i are the minimal primes of R . In particular, $D(R/k)$ is generated as a k -algebra by $\{x^{\mathbf{b}}\partial^{\mathbf{a}} : x^{\mathbf{b}} \in P \text{ or } x^{\mathbf{a}} \notin P \text{ for each minimal prime } P \text{ of } R\}$, and the nonzero operators of this form determine a free basis for $D(R/k)$ as a k -module.

Proof. The last statement follows immediately from the first claim and Theorem 5.2.1. It remains to prove the initial claim.

Suppose that $x^{\mathbf{b}}\partial^{\mathbf{a}} \in D(R)$. Let P be a minimal prime of J . Relabeling, if necessary, $P = (x_1, \dots, x_d)$. If $x^{\mathbf{a}} \in P$ then $x^{\mathbf{a}}x_{d+1} \cdots x_N \in J$. As $x^{\mathbf{b}}\partial^{\mathbf{a}}$ fixes J , $x^{\mathbf{b}}\partial^{\mathbf{a}} * x^{\mathbf{a}}x_{d+1} \cdots x_N = x^{\mathbf{b}}x_{d+1} \cdots x_N \in J \subset P$. As $x_{d+1} \cdots x_N \notin P$ and as P is prime, $x^{\mathbf{b}} \in P$.

Now suppose that $x^{\mathbf{b}}\partial^{\mathbf{a}}$ is such that for each minimal prime P of J , $x^{\mathbf{b}} \in P$ or $x^{\mathbf{a}} \notin P$. We show that $x^{\mathbf{b}}\partial^{\mathbf{a}} \in \mathbb{I}(J)$ (so $x^{\mathbf{b}}\partial^{\mathbf{a}}$ restricts to a differential operator on R). Suppose that $x^{\mathbf{c}} \in J$. For each minimal prime P of J , $x^{\mathbf{c}} \in P$. If $x^{\mathbf{a}} \notin P$ then $\partial^{\mathbf{a}}x^{\mathbf{c}} \in P$ (if $\partial^{\mathbf{a}}x^{\mathbf{c}} \notin P$ then $x^{\mathbf{a}}(\partial^{\mathbf{a}}x^{\mathbf{c}}) = nx^{\mathbf{c}} \in P$ for some $n \in \mathbb{Z}$ and this forces $x^{\mathbf{a}} \in P$, a contradiction). Also, if $x^{\mathbf{b}} \in P$, then $x^{\mathbf{b}}\partial^{\mathbf{a}} * x^{\mathbf{c}} \in P$. Since these results hold for each minimal prime of the radical ideal J , $x^{\mathbf{b}}\partial^{\mathbf{a}} * x^{\mathbf{c}} \in J$. Hence $x^{\mathbf{b}}\partial^{\mathbf{a}} \in \mathbb{I}(J)$ and $x^{\mathbf{b}}\partial^{\mathbf{a}}$ restricts to a differential operator on R . \square

Remark 5.2.4. Theorem 5.2.3 can be applied in practical examples because Proposition 5.1.1 enables us to compute the minimal primes of R . Theorem 5.2.3 also holds when J is not a monomial ideal but becomes a monomial ideal after tensoring with $\text{frac}(k)$; for example, the theorem applies to $R = \frac{\mathbb{Z}[x,y]}{(2x,xy)}$. We omit the details.

There is an alternate statement of the theorem using double annihilators.

Corollary 5.2.5. *Given a reduced monomial ring $R = \frac{k[\mathbf{x}]}{J}$, $x^{\mathbf{b}}\partial^{\mathbf{a}}$ is in $D(R)$ if and only if $x^{\mathbf{b}} \in (J : (J : x^{\mathbf{a}}))$. That is,*

$$D(R) = \bigoplus_{\mathbf{a}} \frac{(J : (J : x^{\mathbf{a}}))}{J} \partial^{\mathbf{a}} \quad (\text{colons here are being computed in } k[\mathbf{x}]).$$

Proof. As J is radical, J is the intersection of its minimal primes, $J = P_1 \cap \cdots \cap P_t \cap P_{t+1} \cap \cdots \cap P_s$. Suppose that $x^{\mathbf{a}} \notin P_i$ ($1 \leq i \leq t$) and $x^{\mathbf{a}} \in P_j$

($j > t$). Then

$$\begin{aligned}
(J : (J : x^{\mathbf{a}})) &= (J : (\bigcap_{i=1}^s P_i : x^{\mathbf{a}})) = (J : \bigcap_{i=1}^s (P_i : x^{\mathbf{a}})) \\
&= (J : \bigcap_{i=1}^t (P_i : x^{\mathbf{a}})) = (J : \bigcap_{i=1}^t P_i) = (\bigcap_{j=1}^s P_j : \bigcap_{i=1}^t P_i) \\
&= \bigcap_{j=1}^s (P_j : \bigcap_{i=1}^t P_i) = \bigcap_{j=t+1}^s (P_j : \bigcap_{i=1}^t P_i) = \bigcap_{j=t+1}^s P_j.
\end{aligned}$$

Now the result follows from Theorem 5.2.3. \square

Corollary 5.2.5 partially extends a result due to Brumatti and Simis [8, Theorem 2.2.1].

Theorem 5.2.6. *Let $J \subset k[\mathbf{x}]$ be an ideal generated by monomials whose exponents are relatively prime to the characteristic of k . If $R = \frac{k[\mathbf{x}]}{J}$ then $Der_k(R) = \bigoplus I_j \frac{\partial}{\partial x_j}$ where $I_j = \frac{(J : (J : x_j))}{J}$.*

Proof. First we show that $Der_k(R)$ is a sum $\sum I_j \frac{\partial}{\partial x_j}$, where the I_j are monomial ideals of R ; then we show that this sum is direct. Let $\theta = \sum_{\mathbf{a}, j} c_{\mathbf{a}, j} x^{\mathbf{a}} \frac{\partial}{\partial x_j}$ be a derivation with each $c_{\mathbf{a}, j} \in k$. Without loss of generality, no term $c_{\mathbf{a}, j} x^{\mathbf{a}} \frac{\partial}{\partial x_j}$ satisfies $x_j | x^{\mathbf{a}}$. If $x^{\mathbf{m}}$ is an arbitrary monomial of J , then all the nonzero terms $c_{\mathbf{a}, j} x^{\mathbf{a}} \frac{\partial}{\partial x_j} * x^{\mathbf{m}}$ are in J , because J is a monomial ideal. This implies that each term $c_{\mathbf{a}, j} x^{\mathbf{a}} \frac{\partial}{\partial x_j}$ induces a derivation on R . This forces each I_j to be a monomial ideal.

Now we show that the sum $\sum I_j \frac{\partial}{\partial x_j}$ is direct. Let $\theta = \sum g_j \frac{\partial}{\partial x_j}$ be a derivation on $k[\mathbf{x}]$ that induces the zero derivation on R . Then for each index j ,

$$g_j = \sum_j g_j \frac{\partial}{\partial x_j} * x_j = \theta * x_j \in J,$$

so each term $g_j \frac{\partial}{\partial x_j}$ of θ also induces the zero operator.

The above considerations did not use the hypothesis on the exponents of the generators of J . Using this hypothesis, the reader can easily check that $I_j = \frac{(J:(J:x_j))}{J}$. \square

5.3 Alternative descriptions of $D(R)$

5.3.1 $D(R)$ in terms of the simplicial complex

The description of the ring of differential operators on a Stanley-Reisner ring can be stated in the language of simplicial complexes. A good reference for this material is Bruns and Herzog [9]. To each Stanley-Reisner ring, we can associate a simplicial complex. Recall that a simplicial complex consists of a collection Δ of finite subsets of a given set V , such that

- (1) for each element $v \in V$, the singleton set $\{v\}$ is in Δ , and
- (2) if $F \subset G \in \Delta$ then $F \in \Delta$.

The elements of V are called vertices and the elements of Δ are called faces. Faces which are maximal with respect to inclusion are called facets. A subcomplex Δ' of Δ is a subset $\Delta' \subseteq \Delta$ such that $F \subset G \in \Delta' \Rightarrow F \in \Delta'$.

If $R = \frac{k[\mathbf{x}]}{J}$ is a Stanley-Reisner ring, let V be the set of variables in $k[\mathbf{x}]$ that are nonzero in R . Identify the nonzero squarefree monomial $x^{\mathbf{b}} \in R$ with the set $\sigma(x^{\mathbf{b}})$ of variables dividing $x^{\mathbf{b}}$. Now define the simplicial complex Δ on the set V to be the smallest simplicial complex satisfying

$$\sigma(x^{\mathbf{b}}) \in \Delta \iff x^{\mathbf{b}} \notin J.$$

For example, the monomial xyz corresponds to the face $\sigma(xyz) = \{x, y, z\} \in \Delta$.

Example 5.3.1. The ring $R = \frac{k[x,y,z,w,u,v]}{(xw,xu,xv,yw,yu,yv,zu,zv)}$ corresponds to the simplicial complex

$$\begin{aligned} \Delta = & \{\emptyset, \{x\}, \{y\}, \{z\}, \{w\}, \{u\}, \{v\}\} \cup \\ & \{\{x, y\}, \{x, z\}, \{y, z\}, \{z, w\}, \{w, u\}, \{w, v\}, \{u, v\}, \{x, y, z\}, \{u, v, w\}\}, \end{aligned}$$

which, in turn, is represented by the diagram in Figure 5.1.

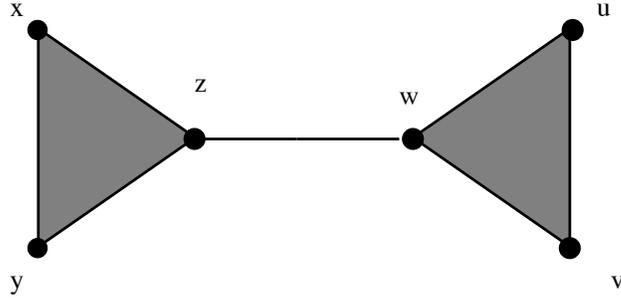


Figure 5.1: Simplicial complex associated to $R = \frac{k[x,y,z,w,u,v]}{(xw,xu,xv,yw,yu,yv,zu,zv)}$.

Here, a line joins two vertices x and y if $\sigma(xy) \in \Delta$ and the region with vertices x , y , and z is shaded to indicate that the face $\sigma(xyz) \in \Delta$. Note that the facets are the sets $\{x, y, z\}$, $\{z, w, u, v\}$ and $\{u, v, w\}$. The minimal primes of R are: (u, v, w) , (x, y, u, v) and (x, y, z) .

Example 5.3.2. The simplicial complex can also have non-trivial homology. For instance, consider $R = \frac{k[x,y,z,w]}{(xz,xw,yzw)}$. The associated simplicial complex appears in Figure 5.2.

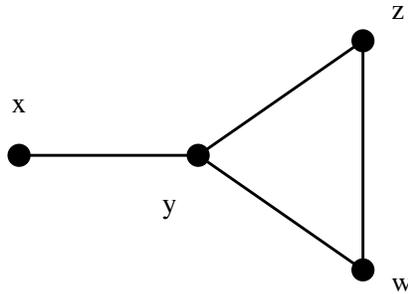


Figure 5.2: Simplicial complex associated to $R = \frac{k[x,y,z,w]}{(xz,xw,yzw)}$.

Here the region with vertices y , z and w is not shaded. This indicates that $\sigma(yz)$, $\sigma(zw)$ and $\sigma(yw)$ are in Δ but $\sigma(yzw)$ is not in Δ .

The facets of this complex are $\{x, y\}$, $\{y, z\}$, $\{y, w\}$ and $\{z, w\}$. The minimal primes of R are: (z, w) , (x, w) , (x, z) and (x, y) .

These two examples suggest that the minimal primes of a Stanley-Reisner ring R are related to the facets of the associated simplicial complex. This gives a more useful way to identify the minimal primes of a Stanley-Reisner ring than Proposition 5.1.1. The following theorem is well-known (see [9, Theorem 5.1.4] for a proof).

Theorem 5.3.3. *The minimal primes of a Stanley-Reisner ring R stand in one-to-one correspondence with the facets of the associated simplicial complex. The variables labeling vertices not occurring in a facet generate a minimal prime of R .*

In this spirit, every radical monomial ideal I of a Stanley-Reisner ring R corresponds to a subcomplex $\Delta(I)$ of the associated complex Δ : $\sigma(x^b)$ is a face of $\Delta(I)$ if and only if $x^b \notin I$. Theorem 5.3.3 says that if P is a minimal prime of R , then $\Delta(P)$ is a facet of the complex Δ associated to R . In Example 5.3.2 the subcomplex σ defined by the ideal $I = (x, zw)$ is shown in Figure 5.3.

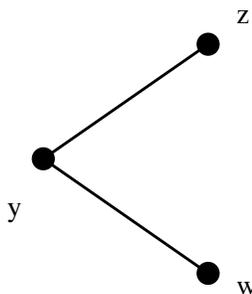


Figure 5.3: Subcomplex determined by the ideal $I = (x, zw)$.

Since the possible coefficients $x^{\mathbf{b}}$ of a monomial operator $x^{\mathbf{b}}\partial^{\mathbf{a}} \in D(R)$ are determined by the minimal primes of R , we can describe the possible coefficients in terms of the facets of the associated simplicial complex.

Theorem 5.3.4. *If R is a Stanley-Reisner ring, then $x^{\mathbf{b}}\partial^{\mathbf{a}} \in D(R)$ if and only if $x^{\mathbf{b}}$ is in the ideal corresponding to the union of the facets not containing all the variables dividing $x^{\mathbf{a}}$.*

Proof. This is essentially a restatement of Theorem 5.2.3 in the language of simplicial complexes. We know that $x^{\mathbf{b}}\partial^{\mathbf{a}} \in D(R)$ if and only if

$$x^{\mathbf{b}} \in \bigcap_{x^{\mathbf{a}} \in P_j} P_j.$$

To be pedantic, this is to say that $x^{\mathbf{b}}$ is in each P_j containing $x^{\mathbf{a}}$. This is equivalent to saying that $\sigma(x^{\mathbf{b}})$ is not contained in each facet $\Delta(P_j)$ that does not contain all the variables dividing $x^{\mathbf{a}}$. So

$$\sigma(x^{\mathbf{b}}) \notin \bigcup \{ \sigma, \tau, \dots \text{ is a facet not containing all variables dividing } x^{\mathbf{a}} \}.$$

This is precisely what it means to say that $x^{\mathbf{b}}$ is in the ideal corresponding to the union of the facets not containing all the variables dividing $x^{\mathbf{a}}$. \square

Example 5.3.5. In the ring treated in Example 5.3.2, $R = \frac{k[x,y,z,w]}{(xz,xw,yzw)}$, the monomial operator $x^{\mathbf{b}}\partial_x^c\partial_y^d$ is in $D(R)$ if and only if

$$\begin{aligned} & x^{\mathbf{b}} \in \text{ideal defined by } \bigcup \{ \text{facets not containing both } x \text{ and } y \} \\ \iff & x^{\mathbf{b}} \in \text{ideal corresponding to subcomplex with facets } \{y,z\}, \{z,w\} \text{ and } \{y,w\} \\ \iff & x^{\mathbf{b}} \in (x). \end{aligned}$$

Similarly, $x^{\mathbf{b}}\partial_x^c\partial_w^d$ ($c, d > 0$) is in $D(R)$ if and only if

$$\begin{aligned} & x^{\mathbf{b}} \in \text{ideal defined by } \bigcup \{ \text{facets not containing both } x \text{ and } w \} \\ \iff & x^{\mathbf{b}} \in \text{ideal corresponding to } \Delta \\ \iff & x^{\mathbf{b}} \in (0). \end{aligned}$$

So $D(R)$ contains no nonzero operators of the form $x^b \partial_x^c \partial_w^d$ with $c, d > 0$.

5.3.2 $D(R)$ and the local cohomology of R

As we saw in chapter 2, $D(R)$ is just the zeroth local cohomology of the $R \otimes_k R$ -module $\text{End}_k(R)$ with support in the kernel of the multiplication map $R \otimes_k R \rightarrow R$. For a Stanley-Reisner ring R , the ring of differential operators $D(R)$ can also be described in terms of the local cohomology of R .

Let $R = \frac{k[x]}{P_1 \cap \dots \cap P_s}$ be a Stanley-Reisner ring with minimal primes P_1, \dots, P_s . For $x^{\mathbf{a}} \in R$, set

$$\Gamma(x^{\mathbf{a}}) = \cap \{P_i : x^{\mathbf{a}} \notin P_i\}.$$

The local cohomology of R with support in $\Gamma(x^{\mathbf{a}})$ has a particularly nice form:

$$H_{\Gamma(x^{\mathbf{a}})}^0(R) = \cap \{P_j : x^{\mathbf{a}} \in P_j\}. \quad (*)$$

To see this, suppose that $x^{\mathbf{a}} \in P_1 \cap \dots \cap P_t \setminus (P_{t+1} \cup \dots \cup P_s)$, so that $\Gamma(x^{\mathbf{a}}) = P_{t+1} \cap \dots \cap P_s$. Now if $y \in P_1 \cap \dots \cap P_t$, then

$$y \Gamma(x^{\mathbf{a}}) \subset (P_1 \cap \dots \cap P_t)(P_{t+1} \cap \dots \cap P_s) \subset P_1 \cap \dots \cap P_s = (0),$$

so $y \in H_{\Gamma(x^{\mathbf{a}})}^0(R)$. Now consider $y \in H_{\Gamma(x^{\mathbf{a}})}^0(R)$. By prime avoidance, there exists a $t \in (P_{t+1} \cap \dots \cap P_s) \setminus (P_1 \cup \dots \cup P_t)$. Because $y \in H_{\Gamma(x^{\mathbf{a}})}^0(R)$, $yt = 0 \in P_1 \cap \dots \cap P_t$, whence $y \in P_1 \cap \dots \cap P_t$.

Now the ring of differential operators on a Stanley-Reisner ring R can be described in terms of the local cohomology of R .

Theorem 5.3.6. *If $R = \frac{k[x]}{P_1 \cap \dots \cap P_s}$ is a Stanley-Reisner ring, then the ring of differential operators on R is*

$$D(R) = \oplus_{\mathbf{a}} H_{\Gamma(x^{\mathbf{a}})}^0(R) \partial^{\mathbf{a}}.$$

Proof. Our remarks above show that

$$H_{\Gamma(x^{\mathbf{a}})}^0(R) = \cap\{P_j : x^{\mathbf{a}} \in P_j\}. \quad (*)$$

By Theorem 5.2.3, $x^{\mathbf{b}}\partial^{\mathbf{a}} \in D(R)$ if and only if $x^{\mathbf{b}} \in \cap\{P_j : x^{\mathbf{a}} \in P_j\}$. Together, these two observations show that $x^{\mathbf{b}}\partial^{\mathbf{a}} \in D(R)$ if and only if $x^{\mathbf{b}} \in H_{\Gamma(x^{\mathbf{a}})}^0(R)$. The sum is clearly direct: if $\theta = \sum_{\mathbf{a}} \gamma(\mathbf{a})\partial^{\mathbf{a}} \in D(R)$ equals zero then let $\gamma(\mathbf{a})\partial^{\mathbf{a}}$ be the nonzero term of smallest total degree in the ∂_i and compute:

$$\gamma(\mathbf{a}) = \theta * x^{\mathbf{a}} = 0.$$

This gives a contradiction and forces all the terms in the representation of θ to be zero. \square

5.4 D-module structure of Stanley-Reisner rings

Having characterized the ring of differential operators on a reduced monomial ring R , we now investigate the D-module structure of R . As we have already seen, monomial rings provide a good source of examples when studying the D-module structure of a general commutative ring.

When $R = \frac{k[x]}{J}$ is a reduced k -algebra and k is a field, then there is a particularly nice description of the D-submodules of R in terms of the minimal primes of R (see Theorem 5.4.5). However, when k is an arbitrary domain, things are more complicated. This is essentially because elements of $D(R/k)$ are k -linear endomorphisms of R so any ideal of R extended from an ideal of k is a $D(R/k)$ -submodule of R .

Example 5.4.1. To begin, we determine which radical monomial ideals of $R = \frac{k[x,y,z]}{(xy,yz)}$ are D-stable. The zero ideal is the intersection of its minimal

primes,

$$(xy, yz) = (x, z) \cap (y).$$

By Theorem 5.2.3, the ring of differential operators, $D(R/k)$ is the R -algebra generated by $\{x\partial_1^n\partial_3^m, z\partial_1^n\partial_3^m, y\partial_2^n\}_{n,m \geq 0}$. The lattice of radical monomial ideals of R consists of the following ideals: $(xy, yz), (y), (xy, yz, xz), (x, yz), (z, xy), (x, z), (z, y), (x, y), (x, y, z)$. Of these, only $(xy, yz), (y), (x, z), (x, y, z)$ are $D(R)$ -submodules of R .

Are there any more D -stable ideals I when we relax the requirement that the ideal I is a radical monomial ideal? The answer turns out to be a qualified no.

Lemma 5.4.2. *In a Stanley-Reisner ring R over an arbitrary domain k , every $D(R)$ -submodule of R , I , is an ideal generated by monomial terms that is equal to a radical monomial ideal after tensoring with $\text{frac}(k)$. In particular, when k is a field, every $D(R)$ -submodule of R is a radical monomial ideal.*

Proof. If f is in some D -stable ideal, I , and $cx^{\mathbf{a}}$ is a term of f of maximal total degree, then $x^{\mathbf{a}}\partial^{\mathbf{a}} \in D(R)$ (by Theorem 5.2.3) and $x^{\mathbf{a}}\partial^{\mathbf{a}} * f = cx^{\mathbf{a}} \in I$. It follows that every term of f is in I and hence I is generated by monomial terms. Again, Theorem 5.2.3 implies that $x_i\partial_i^n \in D(R)$. If $hx^{\mathbf{a}} \in I$ ($h \in k$), then let $\mathbf{d} \in \mathbb{N}^N$ be given by $d_i = 0$ if $a_i = 0$ and $d_i = 1$, if $a_i \neq 0$ (that is, $d_i = 1 - \delta_{a_i, 0}$). Then $x^{\mathbf{d}}\partial^{\mathbf{a}} \in D(R)$ and so $hx^{\mathbf{d}} = x^{\mathbf{d}}\partial^{\mathbf{a}} * hx^{\mathbf{a}} \in I$. If k is a field then $x^{\mathbf{d}} \in I$ and I is radical. \square

Remark 5.4.3. The isolated singularity $R = \frac{\mathbb{C}[x, y, z]}{(x^3 + y^3 + z^3)}$ formed by taking the affine cone over a plane cubic is at the center of some interesting mathematics. In [4] Bernstein, Gelfand and Gelfand describe $D(R)$. In particular,

they show that there are no differential operators in $D(R)$ of negative degree. It follows that m , and all powers of m , are D-stable. Lemma 5.4.2 shows that this behavior does not hold in Stanley-Reisner rings: powers of D -stable ideals are almost never D -stable (this only occurs for ideals of R which are extended from k). Also, this example shows that Lemma 5.4.2 does not extend to arbitrary algebras over a field.

At present, there is no nice description of $D(R)$ when \mathbb{C} is replaced by a field of characteristic p . It is known that the description of $D(R)$ found in [4] does not extend to the characteristic p case (there are strictly more differential operators in the characteristic p case: see Smith [47, page 385] for a precise statement of this fact); however, it is not known whether there are any negative degree differential operators in the characteristic p case. The existence of such an operator would have implications for the theory of tight closure. For a detailed study of the theory of tight closure in this example, see McDermott [34].

Example 5.4.4. We investigate the D-module structure of R more closely. Let $R = \frac{k[x,y,z,w]}{(xz,yw,yz)}$, k a field. The zero ideal is the intersection of its minimal primes,

$$(xz, yw, yz) = (x, y) \cap (y, z) \cap (w, z).$$

The R -algebra $D(R)$ is generated by $\{x\partial_1^n, y\partial_1^n\partial_2^m, z\partial_3^n\partial_4^m, w\partial_4^n\}_{n,m \geq 0}$. There are 34 radical monomial ideals in R and among these, 14 are D-stable (see Table 5.1). The 14 D-stable ideals are all ideals which can be obtained by taking sums and intersections of the minimal primes of R .

This suggests the following result.

Theorem 5.4.5. *When k is a field, the D-submodules of a reduced monomial ring $R = \frac{k[x]}{\mathcal{J}}$ are precisely the intersections of sums of minimal primes*

Reference Number	Ideal	D(R)-stable?	Representation of Ideal
1	$(0) = (xz, yw, yz)$	Yes	$P \cap Q \cap L$
2	(x, yw, yz)	No	
3	(x, y)	Yes	P
4	(x, y, zw)	No	
5	(x, y, z)	Yes	$P + Q$
6	(x, y, w)	No	
7	(x, y, z, w)	Yes	$P + Q + L$
8	(x, z, yw)	No	
9	(x, z, w)	No	
10	(x, w, yz)	No	
11	(y, xz)	Yes	$P \cap Q$
12	(y, z)	Yes	Q
13	(y, z, w)	Yes	$Q + L$
14	(y, z, xw)	Yes	$(P + Q) \cap (Q + L)$
15	(y, w, xz)	No	
16	(y, xw, xz)	Yes	$P \cap (Q + L)$
17	(y, xw, zw, xz)	No	
18	(y, zw, xz)	No	
19	(z, yw)	Yes	$Q \cap L$
20	(z, w)	Yes	L
21	(z, w, xy)	No	
22	(z, xy, yw)	No	
23	(z, xy, xw, yw)	No	
24	(z, xw, yw)	Yes	$(P + Q) \cap L$
25	(w, xz, yz)	No	
26	(w, xy, xz, yz)	No	
27	(xy, xz, yw, yz)	No	
28	(xy, xw, xz, yw, yz)	No	
29	(xy, xw, zw, xz, yw, yz)	No	
30	(xy, zw, xz, yw, yz)	No	
31	(xw, xz, yw, yz)	Yes	$P \cap L$
32	(xw, zw, xz, yw, yz)	No	
33	(zw, xz, yw, yz)	No	
34	1	Yes	1

Table 5.1: $D(R)$ -stability of Radical Ideals of $R = \frac{k[x,y,z,w]}{(xz,yw,yz)}$.

of R .

Proof. As $(0) \subset R$ is D-stable, and R is reduced, Theorem 4.2.1 shows that all minimal primes of R are D-stable. Clearly, sums and intersections of D-stable ideals are D-stable. So intersections of sums of minimal primes are D-stable.

Conversely, let I be a D-stable ideal of R . Then I is a radical monomial ideal by Lemma 5.4.2. Write I as the intersection of its minimal primes. Each minimal prime is D-stable (by Theorem 4.2.1) so it suffices to establish the result for I a prime ideal.

Consider the D-stable prime ideal I as an ideal of $k[\mathbf{x}]$ and note that $I \supset J = P_1 \cap \dots \cap P_r$. Relabeling if necessary, we may assume that $I \supset P_1, \dots, I \supset P_t, I \not\supset P_{t+1}, \dots, I \not\supset P_r$. Note that I must contain at least one minimal prime of J since I contains J and is prime itself; so we have $t \geq 1$.

We claim that $I = P_1 + \dots + P_t$. By hypothesis, $I \supset P_1 + \dots + P_t$. Assume that $I \neq P_1 + \dots + P_t$; we aim to produce an operator $x^{\mathbf{b}}\partial^{\mathbf{a}}$ in $\mathbb{I}(J)$ and not in $\mathbb{I}(I)$. Then $x^{\mathbf{b}}\partial^{\mathbf{a}}$ restricts to an element of $D(R)$ which does not stabilize I . This will be a contradiction as I is $D(R)$ -stable.

As $I \not\supset P_i$ ($i \geq t + 1$), there is a monomial in P_i which is not in I . The product of such monomials is a monomial $x^{\mathbf{b}}$ which is in each P_i ($i \geq t + 1$) and not in I . As $I \neq P_1 + \dots + P_t$, and I is a monomial ideal, there is a monomial $x^{\mathbf{a}} \in I$ with $x^{\mathbf{a}} \notin P_1, \dots, x^{\mathbf{a}} \notin P_t$.

Using the criterion of Theorem 5.2.3, one checks that $x^{\mathbf{b}}\partial^{\mathbf{a}} \in \mathbb{I}(J)$. As I is prime, Theorem 5.2.3 also shows that $x^{\mathbf{b}}\partial^{\mathbf{a}} \in \mathbb{I}(I)$ if and only if $x^{\mathbf{a}} \notin I$ or $x^{\mathbf{b}} \in I$. As neither of these conditions are satisfied, $x^{\mathbf{b}}\partial^{\mathbf{a}} \notin \mathbb{I}(I)$. This shows that I could not have been D-stable, so $I = P_1 + \dots + P_t$ after all. \square

Remark 5.4.6. In fact, the collection of D-ideals equals the lattice of ideals

generated by the minimal primes of R under the operations of addition and intersection. However, this does not improve the theorem: any ideal in this lattice is the intersection of sums of minimal primes of R .

Remark 5.4.7. As expected, Theorem 5.4.5 must be modified when k is not a field. For example, $(0) \subset R = \frac{\mathbb{Z}[x]}{(4x)}$ is clearly D -stable, but (0) is not the intersection of sums of minimal primes of R . Still, this becomes true after tensoring with $\mathbb{Q} = \text{frac}(\mathbb{Z})$. In general, when k is a commutative domain, the $D(R)$ -submodules of R become the intersection of sums of minimal primes of $R \otimes_k \text{frac}(k)$ after tensoring with $\text{frac}(k)$.

The D -module structure of a Stanley-Reisner ring can be used to give an alternative proof of Theorem 5.2.3. In fact all that is necessary is Theorem 4.2.1, concerning the stability of minimal components of stable ideals. This second proof is similar to Tripp's argument [55], which also involves an idealizer computation.

Theorem 5.2.3. *If $R = \frac{k[x_1, \dots, x_n]}{J}$ is a Stanley-Reisner ring, then an element of the Weyl algebra $\theta = \sum_{\alpha, \beta} c_{\alpha, \beta} x^\alpha \partial^\beta$ ($\partial_i^n = \frac{1}{n!} \frac{\partial^n}{\partial x_i^n}$) defines a differential operator on R if and only if for each minimal prime P of J , and each nonzero term $c_{\alpha, \beta} x^\alpha \partial^\beta$ of θ , either $x^\beta \notin P$ or $x^\alpha \in P$. Such terms $x^\alpha \partial^\beta$ generate the R -algebra $D(R)$.*

Proof. Write J as an intersection of primes, $J = P_1 \cap \dots \cap P_s$, where each prime P_i is generated by a subset of the variables. Consider a general primary component $P = (x_1, \dots, x_t)$ of J . Let θ be an element of the idealizer of J representing a differential operator on R . Because differential operators on R stabilize the minimal primary components of the zero ideal of R (Theorem 4.2.1), $\theta \in \mathbb{I}(P) + JW = \mathbb{I}(P)$. Now θ is in $\mathbb{I}(P) = \mathbb{I}(x_1, \dots, x_t)$ if

and only if

$$\theta \in k[x_1, \dots, x_n] \langle \partial_{t+1}^m, \dots, \partial_n^m, x_1 \partial_1^{n_1} \dots \partial_t^{n_t}, \dots, x_t \partial_1^{n_1} \dots \partial_t^{n_t} \rangle_{m, n_1, \dots, n_t \in \mathbb{N}}.$$

This is clearly equivalent to saying that $\theta = \sum_{\alpha, \beta} c_{\alpha, \beta} x^\alpha \partial^\beta$ where either $x^\beta \notin P$ or $x^\alpha \in P$. Because θ stabilizes each minimal prime, this must hold for all the minimal primes P_i of R .

Conversely, if an element θ of the Weyl algebra is in each idealizer $\mathbb{I}(P_i)$, then

$$\theta \in \mathbb{I}(P_1) \cap \dots \cap \mathbb{I}(P_s) \subseteq \mathbb{I}(P_1 \cap \dots \cap P_s) = \mathbb{I}(J)$$

and so θ represents a differential operator, $\theta \in D(R)$. Now the result follows from the characterization of $\mathbb{I}(x_1, \dots, x_t)$ described above. \square

Chapter 6

Nakai's Conjecture

Nakai's conjecture relates the algebraic structure of the ring of differential operators on a \mathbb{C} -algebra R to the presence of singularities in $\text{Spec}(R)$. Ishibashi [23] proposed an analogue of Nakai's conjecture in prime characteristic. Nakai's conjecture would imply the Zariski-Lipman conjecture relating smoothness to the module of derivations. These conjectures are introduced in the first section. The description of D-stable ideals in Stanley-Reisner rings provided by Theorem 5.4.5 gives an easy proof that Nakai's conjecture holds for Stanley-Reisner rings of arbitrary characteristic. In the second section, Corollary 4.3.3 on the HS-stability of the conductor ideal is used to establish Nakai's conjecture (in arbitrary characteristic) for varieties whose normalization is smooth. The last section uses this theorem to extend a theorem due to Brown: certain hypotheses on the D-module structure of R are sufficient to imply that R is normal.

6.1 Introducing Nakai’s conjecture

Nakai, inspired by Grothendieck’s observation that the ring of differential operators on a smooth complex variety is generated by derivations, conjectured the converse: the ring of differential operators on a complex variety is generated by derivations if and only if the variety is nonsingular. Nakai appears not to have stated the conjecture in the literature but it is often quoted in connection with his paper [38]. The first statement of the conjecture appears in [37]. Ishibashi [23] restated the conjecture in a characteristic-free manner by using the Hasse-Schmidt derivations: an affine variety $\text{Spec}(R)$ defined over an algebraically closed field k is nonsingular if and only if the ring of differential operators $D(R/k)$ equals the Hasse-Schmidt algebra, $HS(R/k)$.

We observed in Theorem 3.3.1 that Grothendieck’s result does not require that the field k be algebraically closed. As well, since the ring of differential operators on R over k is a relative object, it makes more sense to relate the structure of $D(R/k)$ to the smoothness of the extension $k \rightarrow R$. This leads to the following more general statement of the conjecture.

Nakai’s Conjecture: Let $X = \text{Spec}(R)$ be a variety defined over a field k ; then X is smooth over k if and only if $D(R/k) = HS(R/k)$.

The Zariski-Lipman conjecture (treated in Lipman [29]) pre-dates Nakai’s conjecture; it also relates the algebraic structure of a differential object to the presence of singularities.

The Zariski-Lipman Conjecture: A complex variety $X = \text{Spec}(R)$ is smooth at a closed point x (corresponding to a maximal ideal $m \subset R$) if and only if the module of derivations $Der(R_m/\mathbb{C})$ is a free R_m -module.

Becker and Rego [1] (see also Hochster [15]) proved that Nakai’s con-

ture implies the Zariski-Lipman conjecture.

Theorem 6.1.1. *Let $X = \text{Spec}(R)$ be a complex variety satisfying Nakai's conjecture. Then $\text{Der}(R/\mathbb{C})$ is a locally free R -module if and only if X is smooth.*

Proof. When X is smooth, $\Omega_{R/\mathbb{C}}$ is a locally free R -module and so $\Omega_{R_P/\mathbb{C}} = \Omega_{R/\mathbb{C}} \otimes_R R_P$ is a free R_P -module for all primes P of R . Then $\text{Der}(R/\mathbb{C}) \otimes_R R_P = \text{Der}(R_P/\mathbb{C}) = \text{Hom}_{R_P}(\Omega_{R_P/\mathbb{C}}, R_P)$ is a free R_P -module for all primes P of R .

Conversely, suppose that $\text{Der}(R/\mathbb{C})$ is a locally free R -module. Lipman [29] has shown that when $\text{Der}(R_P/\mathbb{C})$ is free, then R_P is normal, so R is normal and is a product of normal domains: $R = R_1 \times \cdots \times R_t$. Because

$$\text{Der}((R_1 \times \cdots \times R_t)/\mathbb{C}) \cong \text{Der}(R_1/\mathbb{C}) \times \cdots \times \text{Der}(R_t/\mathbb{C})$$

and

$$D((R_1 \times \cdots \times R_t)/\mathbb{C}) \cong D(R_1/\mathbb{C}) \times \cdots \times D(R_t/\mathbb{C}),$$

it suffices to show that if $\text{Der}(R_i/\mathbb{C})$ is locally free then $D(R_i/\mathbb{C}) = \text{der}(R_i/\mathbb{C})$. In this case, $D(R/\mathbb{C}) = \text{der}(R/\mathbb{C})$, and R is smooth by Nakai's conjecture. So we may assume that R is a normal domain.

Let D_1, \dots, D_r be a set of free R -module generators for $\text{Der}(R/\mathbb{C})$. Because R is normal, every prime ideal P of height 1 is regular (that is, R_P is regular), and

$$R = \bigcap_{\text{ht}P=1} R_P.$$

We show that the monomials $\{D_1^{a_1} \cdots D_r^{a_r}\}_{|a| \leq n}$ of degree $\leq n$ in the derivations D_1, \dots, D_r induce a free basis of the R_P -module $\text{der}_n(R_P/\mathbb{C})$. Since there are $\binom{n+r}{r}$ monomials and they generate $\text{der}_n(R_P/\mathbb{C})$, it will suffice to

show that $der_n(R_P/\mathbb{C})$ has rank $\binom{n+r}{r}$ (in which case these operators are linearly independent and form a basis).

Because R_P is regular, we can find a maximal ideal m containing P such that R_m is regular. Moreover, by adjoining to R the inverse of a suitable minor of the Jacobian matrix, we see that R_m and R_P are localizations of a smooth k -algebra of finite type. Now by Theorem 3.1.9, we can find a regular system of parameters x_1, \dots, x_r for R_m , such that $k[x_1, \dots, x_r] \hookrightarrow R_m$ is formally étale. Then the operators $d_i = \frac{\partial}{\partial x_i}$ induce derivations on R_m that generate $Der(R_m/\mathbb{C})$. In fact, the operators $\{d^{\mathbf{a}} = d_1^{a_1} \cdots d_r^{a_r}\}_{|\mathbf{a}| \leq n}$ induce generators of $der(R_m/\mathbb{C})$. These operators are linearly independent since if

$$\sum r_{\mathbf{a}} d^{\mathbf{a}} = 0,$$

then

$$0 = \sum r_{\mathbf{a}} d^{\mathbf{a}} * x^{\mathbf{a}} = \mathbf{a}! r_{\mathbf{a}}$$

and all the coefficients in the relation are zero. It follows that $der_n(R_m/\mathbb{C})$ is a free R_m -module of rank $\binom{n+r}{r}$. Tensoring with R_P , we see that $der_n(R_P/\mathbb{C}) = der_n(R_m/\mathbb{C}) \otimes_{R_m} R_P$ is a free R_P -module of the same rank. As well, by Corollary 3.3.2 and Remark 3.3.3, $D_n(R_P/\mathbb{C}) = der_n(R_P/\mathbb{C})$.

Now we show that $D(R/\mathbb{C}) = der(R/\mathbb{C})$. Let D be a differential operator of order $\leq n$ on R . Then

$$\begin{aligned} D \in \bigcap_{ht P=1} D_n(R_P) &= \bigcap_{ht P=1} der_n(R_P) \\ &= \bigcap_{ht P=1} (\bigoplus_{|\mathbf{a}| \leq n} R_P \mathbf{D}^{\mathbf{a}}) \\ &= \bigoplus_{|\mathbf{a}| \leq n} (\bigcap_{ht P=1} R_P) \mathbf{D}^{\mathbf{a}} \\ &= \bigoplus_{|\mathbf{a}| \leq n} R \mathbf{D}^{\mathbf{a}} = der_n(R). \end{aligned}$$

It follows that $der(R) = D(R)$ and so R is smooth (by hypothesis). \square

An easy argument involving D-stable ideals recovers and extends Schreiner's result [41] that Stanley-Reisner rings satisfy Nakai's conjecture in characteristic zero.

Theorem 6.1.2. *When R is a Stanley-Reisner ring defined over a field k , $D(R/k) = HS(R/k)$ if and only if R is a polynomial ring.*

Proof. When R is a polynomial ring, Theorem 3.3.1 forces $D(R/k) = HS(R/k)$. For the other implication, suppose that $D(R/k) = HS(R/k)$ and P is a minimal prime of R . By Theorem 4.2.4, P is HS -stable, and by Theorem 4.1.2, P^2 is HS -stable. Because $D(R/k) = HS(R/k)$, P^2 is D-stable. Lemma 5.4.2 forces $P^2 = 0$ and since R is reduced, $P = 0$. So R is a domain. Because a Stanley-Reisner ring which is a domain is a polynomial ring, this completes the proof. \square

6.2 Nakai's conjecture for varieties with smooth normalization

In this section we prove Nakai's conjecture for varieties whose normalization is smooth. To begin, we treat the case of a cusp.

Example 6.2.1. The differential operators on the cusp singularity $y^2 = x^3$ are not generated by derivations. Consider the coordinate ring:

$$R = \frac{\mathbb{C}[x, y]}{(y^2 - x^3)} \cong \mathbb{C}[t^2, t^3].$$

Its normalization R' is the polynomial ring $\mathbb{C}[t]$, which is smooth over \mathbb{C} .

The conductor of R' into R is $C = (t^2, t^3)R$. Note that $C^2 = (t^4, t^5)R$.

It is easy to check that the derivations on R , $Der(R)$, are generated by $t \frac{d}{dt}$ and $t^2 \frac{d}{dt}$. Since these do not lower t -adic order, both C and C^2 are

$\text{der}(R)$ -stable.

Consider, the differential operator $\gamma = \frac{d^4}{dt^4}$ on $R' = \mathbb{C}[t]$. The operator γ sends $t^4 \in C^2$ to a unit. Now multiply γ by the element t^2 (in the conductor C) to get $t^2\gamma$, a differential operator on R . The operator $t^2\gamma$ does not stabilize C^2 : t^4 is sent to $24t^2$.

Since C^2 is $\text{der}(R)$ -stable, but not $D(R)$ -stable, $D(R)$ is not generated by derivations.

Most of the ideas used in this example reappear in the general situation. Before treating the general case, we need one technical lemma concerning conductor ideals on reduced rings.

Lemma 6.2.2. *If R is a reduced ring and R' is the integral closure of R in its total ring of quotients, then the conductor C of R' into R is not contained in any minimal prime of R .*

Proof. The conductor C equals $\text{Ann}_R(\frac{R'}{R})$. If $C \subseteq P$ with P a minimal prime of R then, since $\text{Supp}(\frac{R'}{R}) = \mathbb{V}(\text{Ann}(\frac{R'}{R}))$, $P \in \text{Supp}(\frac{R'}{R})$. Localizing the exact sequence of R -modules

$$0 \rightarrow R \rightarrow R' \rightarrow \frac{R'}{R} \rightarrow 0$$

at the minimal prime P gives an exact sequence

$$0 \rightarrow R_P \rightarrow R'_P \rightarrow (\frac{R'}{R})_P \rightarrow 0.$$

But since P is a minimal prime of a reduced ring, R_P is a field and the normalization map $R_P \rightarrow R'_P$ is an isomorphism. This forces $(\frac{R'}{R})_P = 0$, a contradiction. So C is not contained in any minimal prime of R . \square

Theorem 6.2.3. *Let R be a reduced k -algebra and let R' be its integral closure in its total ring of quotients. If R' is a product of D -simple rings and $HS(R/k) = D(R/k)$, then R is normal.*

Proof. Note that R' is isomorphic to $R_1 \times \cdots \times R_t$, where R_i is the normalization of $\frac{R}{P_i}$ with the $\{P_i\}$ ranging over the minimal primes of R . By Corollary 4.3.3, C is $HS(R/k)$ -stable and by Theorem 4.1.2, C^2 is also $HS(R/k)$ -stable. Thus, C^2 is $D(R/k)$ -stable.

Assume that $C^2 \neq C$. Since C is not contained in any of the minimal primes P_i of R , there are elements $c_i \in C \setminus P_i$ with some $c_j \notin C^2$. To see this, take $x \in C \setminus C^2$ and note that $x \neq 0 \Rightarrow x \notin \cap P_i \Rightarrow x$ is not in some P_j . Set $c_j = x$ and pick the other $c_i \in C \setminus P_i$. Let $c = (c_1, \dots, c_t) \in R'$, after identifying R' with the product in the first paragraph. Then $c \in C \setminus C^2$ and c is nonzero in each component. By the D-simplicity of each of the R_i , there is an operator $\theta = (\theta_1, \dots, \theta_t) \in D(R_1) \times \cdots \times D(R_t)$, such that $\theta(c^2) = 1$. If each $\theta_i \in D(R_i)$ is an operator of order $\leq n_i$ then $\theta = (\theta_1, \dots, \theta_t)$ maps R' to itself and θ is a differential operator of order $\leq n = \max(n_i)$. Thus, $\theta \in D(R')$ and $c\theta \in D(R)$. Now $(c\theta)(c^2) = c \notin C^2$, contradicting the fact that C^2 is $D(R/k)$ -stable. This forces $C^2 = C$.

In fact, $C = R$. Indeed, if C is contained in a maximal ideal m , then $C_m = C_m^2 \subset mC_m \subset C_m$, so $mC_m = C_m$. Now Nakayama's lemma forces $C_m = 0$. But then C must consist of zero divisors. Using prime avoidance,

C is contained in the union of the associated primes
 $\implies C$ is contained in some associated prime
 $\implies C$ is contained in some minimal prime
 (as R is reduced).

This contradicts Lemma 6.2.2. So $C = R$ and R is normal. □

Theorem 6.2.4. *Let R be a reduced algebra over a field k . If $HS(R/k) = D(R/k)$ and the normalization R' of R is smooth over k , then R is smooth over k .*

Proof. Since the normalization R' of R is product of smooth domains, R' is a product of D -simple rings (by Corollary 3.3.4). The result now follows from Theorem 6.2.3. \square

Together with Theorem 3.3.1, Theorem 6.2.4 shows that Nakai's conjecture holds for reduced varieties smoothed by normalization; for example, Nakai's conjecture holds for curves. This rederives and extends a result of Mount and Villamayor in characteristic zero [37]. Ishibashi [23] announced this result in prime characteristic but his proof contains an unfixable error.

Corollary 6.2.5. *Let R be a reduced algebra over a field k . If R is 1-dimensional and $HS(R/k) = D(R/k)$ then R is smooth over k . In particular, if R is a domain of characteristic 0 and $der(R/k) = D(R/k)$ then R is regular.*

Theorem 6.2.4 can also be used to give another proof that Nakai's conjecture holds for Stanley-Reisner rings. More generally, the theorem implies that Nakai's conjecture holds for varieties all of whose components are smooth.

Corollary 6.2.6. *Let R be a reduced algebra over a field k . If $HS(R/k) = D(R/k)$ and if $\frac{R}{P}$ is smooth over k for each minimal prime P of R , then R is smooth over k . In particular, if $X = \text{Spec}(R)$ is a hyperplane arrangement and $HS(R/k) = D(R/k)$ then X is smooth over k .*

Both Corollary 6.2.5 and Corollary 6.2.6 follow immediately from Theorem 6.2.4: just observe that the normalization of $X = \text{Spec}(R)$ is isomorphic to the disjoint union of the normalization of the components of X , each of which is smooth over k .

Theorem 6.2.4 says that the ring of differential operators on a singular complex variety whose normalization is smooth is not generated by derivations. Thus, $D(R/\mathbb{C})$ is complicated even for very mild singularities (those that can be resolved by normalization). We expect that $D(R/\mathbb{C})$ will become more complicated as the singularities become more difficult to resolve. This provides further evidence for Nakai's conjecture.

6.3 Extension of Brown's result

When R is smooth over a perfect field k , then R is D-simple (Corollary 3.3.4) but not conversely. For instance, if k is a perfect field of prime characteristic, a strongly F-regular k -algebra of finite type that is not regular provides a counterexample (using the fact that strongly F-regular rings are D-simple – Theorem 4.4.1). An example of such a ring is $R = \frac{k[x_1, \dots, x_n]}{(x_1^2 + \dots + x_n^2)}$, where k is a perfect field of characteristic $p > 2$. Since R is Gorenstein, strong F -regularity is equivalent to weak F -regularity (see Hochster and Huneke [17, Theorem 5.5]). From the proof of Theorem 4.6 in [18], R is weakly F -regular if a homogeneous system of parameters is tightly closed. Using a result due to Smith [48, Theorem 2.7], the tight closure of the system of parameters (x_2, \dots, x_n) does not contain x_1 . It follows that $(x_2, \dots, x_n)^* = (x_2, \dots, x_n)$ and R is a strongly F-regular ring which is not regular (and hence, not smooth over k).

Even though D-simplicity is insufficient to imply that R is smooth, the D-module structure of R ought to have some bearing on the algebraic structure of R . Brown showed that in the presence of the Nakai hypothesis, $D(R/k) = \text{der}(R/k)$, certain D-module conditions imply that R is normal.

Theorem 6.3.1 (Brown [6]). *Let R be a finitely generated integral domain over a field k of characteristic zero such that $D(R/k) = \text{der}(R/k)$. If every D -stable prime ideal of R has height less than or equal to 1, then R is normal.*

This theorem admits an extension to prime characteristic.

Theorem 6.3.2. *Let R be a finitely generated reduced algebra over a field k and suppose that:*

- (1) $D(R/k) = HS(R/k)$ and
- (2) every D -stable prime ideal of R has height less than or equal to 1.

Then R is normal.

Proof. Let C be the conductor of the normalization R' of R into R . By Corollary 4.3.3, C is $HS(R)$ -stable. Then, by Lemma 6.2.2, C is not contained in any minimal prime of R and, by Theorem 4.2.4, the minimal primes of C are $HS(R)$ -stable. Using (1), any minimal prime P of C is D -stable and, using (2), P has height 1.

The Hasse-Schmidt algebra $HS(R_P)$ is contained in $D(R_P)$ and contains the algebra H generated by the components of extensions of Hasse-Schmidt derivations on R to R_P . Since $D(R_P) \cong D(R) \otimes_R R_P = HS(R) \otimes_R R_P$, $D(R_P) = H$ and so $D(R_P) = HS(R_P)$. The ring R_P is 1 dimensional, so its normalization is smooth. By Corollary 6.2.5, R_P is smooth over k . Since k is a field, R_P is regular and $R_P = R'_P$. Then $C_P = R_P$ for any minimal prime P of C . But this is a contradiction since it implies that there exists an element $x \in R \setminus P$ such that $xC \not\subseteq P$. Thus C has no minimal primes: $C = R$ and R is normal. \square

In the same paper (Brown [6]), Brown shows that the hypotheses in Theorem 6.3.1 actually imply that R is regular. His proof uses Zariski's lemma

on derivations (if a derivation on a local ring fails to stabilize the maximal ideal, then the associated variety is analytically a product). Unfortunately, the current prime characteristic analogues of Zariski's lemma (see Brown and Kuan [7] and Ishibashi [22]) are not powerful enough to extend this result to the prime characteristic case.

Chapter 7

Tight Closure

For what follows, we restrict to the case where k is a field of characteristic p . We will define the necessary tight closure terminology but we refer the reader to Hochster and Huneke [16] for details (also, a nice exposition of both the characteristic p and the characteristic zero theory can be found in the notes from the CBMS conference [21]).

When R is a ring of characteristic $p > 0$, R can be viewed as an algebra over its ring of p^e -th powers, R^{p^e} . Recall that the ring R is F-finite if R is a finitely generated module over R^p . When R is F-finite, the description of $D(R/\mathbb{Z})$ in terms of the Frobenius powers of R ,

$$D(R/\mathbb{Z}) = \cup_e \text{End}_{R^{p^e}}(R)$$

suggests that tight closure is connected with the theory of differential operators in characteristic p .

7.1 $D(R)$ and tight closure for Stanley-Reisner rings

If the map $R^p \hookrightarrow R$ splits then R is said to be F-split. If for each R^p -module M , $R^p \otimes M \rightarrow R \otimes M$ is injective, we say that R is F-pure. Note that F-split implies F-purity. In general, purity is equivalent to splitting for finite maps (see Matsumura [32, Theorem 7.14]), so F-split implies F-pure when R is F-finite.

Using differential operators, we obtain an easy proof of a result due to Hochster and Roberts [20, Proposition 5.38] that Stanley-Reisner rings in prime characteristic are F-pure. In fact, we do more: we give an explicit splitting of $R^p \hookrightarrow R$.

Theorem 7.1.1. *If $R = \frac{k[x]}{J}$ is a Stanley-Reisner ring of prime characteristic, then the inclusion map $R^p \hookrightarrow R$ is split by the differential operator $\partial^{(p-1, \dots, p-1)}_{x^{(p-1, \dots, p-1)}}$.*

Proof. Theorem 5.2.3 implies that

$$\partial^{(p-1, \dots, p-1)}_{x^{(p-1, \dots, p-1)}} = \sum_{(i_1, \dots, i_N)=\mathbf{0}}^{(p-1, \dots, p-1)} \prod_{k=1}^N \binom{p-1}{i_k} x_k^{i_k} \partial_k^{i_k}$$

is a differential operator on R . This operator clearly sends 1 to 1. Also, its image is contained in R^p :

$$\begin{aligned} \partial^{(p-1, \dots, p-1)}_{x^{(p-1, \dots, p-1)}} * x^{\mathbf{a}} &= \binom{a_1 + p - 1}{p - 1} \dots \binom{a_N + p - 1}{p - 1} x^{\mathbf{a}} \\ &= \begin{cases} 0 & \text{if } x^{\mathbf{a}} \notin R^p \\ x^{\mathbf{a}} & \text{if } x^{\mathbf{a}} \in R^p \end{cases}, \end{aligned}$$

the last equality coming from Lemma 2.1.10. This computation also shows that this operator is R^p -linear. Thus, this is an R^{p^e} -module splitting of $R^p \hookrightarrow R$. As R is F-split, it is also F-pure. \square

When $I = (a_1, \dots, a_t)$ is an ideal in a Noetherian ring R of characteristic $p > 0$, define $I^{[p^e]}$ to be the ideal $(a_1^{p^e}, \dots, a_t^{p^e})$. The ideal $I^{[p^e]}$ is independent of the choice of generators of I : $I^{[p^e]} = (\{r^{p^e} : r \in I\})$. Let R^0 denote the set of elements of R that are not in any minimal prime of R . To each ideal I we will associate a (possibly) larger ideal I^* , the tight closure of I .

Definition 7.1.2. An element $x \in R$ is in I^* if and only if there exists an element c , not in any minimal prime of R , such that $cx^{p^e} \in I^{[p^e]}$ for all $e \gg 0$. When it is necessary to indicate the ring R , we use the notation I_R^* for the tight closure of I in R .

If R is reduced, the condition *for all* $e \gg 0$ can be replaced with *for all* e . When every ideal I of R is tightly closed, $I^* = I$, we say that R is weakly F-regular. When R and all its localizations are weakly F-regular, then R is said to be F-regular. The notation here is unfortunate but is currently necessary because we do not know when tight closure commutes with localization. Regular rings are F-regular (see [16]).

Tight closure admits a particularly nice description in Stanley-Reisner rings.

Lemma 7.1.3. *The tight closure of an ideal I of a localization of a Stanley-Reisner ring $R = \frac{k[\mathbf{x}]_S}{P_1 \cap \dots \cap P_r}$ is $I_R^* = \bigcap_i (I + P_i)$.*

Proof. Tight closure can be tested modulo minimal primes [21, Theorem 1.3.c]. Thus, $I_R^* = \bigcap_i ((I + P_i)_{\frac{R}{P_i}}^* \cap R)$. But $\frac{R}{P_i} = S^{-1}(\frac{k[\mathbf{x}]}{P_i})$ is a localization of a polynomial ring and, as such, is F-regular. Thus $(I + P_i)_{\frac{R}{P_i}}^* = (I + P_i)_{\frac{R}{P_i}}$. The result follows. \square

As a result, tight closure commutes with localization in Stanley-Reisner rings. It is worth pointing this out in light of the difficulty encountered in

proving this in general. For different perspectives on this result, see Smith and Swanson [50] and Katzman [27]. This result has been extended to a more general class of rings (including the coordinate ring of any variety defined by binomial equations) by Smith [46].

Corollary 7.1.4. *Tight closure commutes with localization in a Stanley-Reisner ring, $R = \frac{k[\mathbf{x}]}{P_1 \cap \dots \cap P_r}$.*

Proof. Let S be a multiplicatively closed set in R . Without loss of generality, P_1, \dots, P_t ($t \leq r$) are the minimal primes of $S^{-1}R$. Let I be an ideal of R . Then Lemma 7.1.3 gives $(IS^{-1}R)^* = \bigcap_{i=1}^t (IS^{-1}R + P_i S^{-1}R) = \bigcap_{i=1}^t [(I + P_i)S^{-1}R] = \bigcap_{i=1}^t [(I + P_i)S^{-1}R] = [\bigcap_{i=1}^t (I + P_i)]S^{-1}R = I_R^* S^{-1}R$. \square

The test ideal of R is the ideal $\{c \in R : cI^* \subset I \text{ for all ideals } I \text{ of } R\}$. A test element of R is an element of the test ideal which is not in any minimal prime of R . Of course, the importance of test elements is that they allow one to identify a particular c for use in all tight closure tests. For details on the existence and applications of test elements, see Hochster and Huneke [16, 17].

A weak test element is an element of R not in any minimal prime of R such that

$$c(I^*)^{[p^e]} \subseteq I^{[p^e]}$$

for all ideals I and all p^e greater than some fixed integer Q (depending only on c and not on the ideal I). If R is F-split, then weak test elements are also test elements:

$$c(I^*)^{[p^e]} \subseteq I^{[p^e]} \Rightarrow (cI^*)^{[p^e]} \subseteq I^{[p^e]} \Rightarrow cI^* \subseteq I.$$

The weak test ideal is the ideal generated by all weak test elements. Because Stanley-Reisner rings are F-split (Lemma 7.1.1), the test ideal of a Stanley-

Reisner ring R equals the weak test ideal of R . Our next goal is to describe this test ideal.

Lemma 7.1.5. *If tight closure in R commutes with localization by an element c in the test ideal, then R_c is F -regular.*

Proof. As tight closure commutes with localization at c , we have $(IR_c)^* = I^*R_c$. If $\frac{z}{1} \in I^*R_c$ then for some s , $c^s z \in I^*$. As c is in the test ideal, $c^{s+1}z \in I$ and hence $z \in IR_c$. The other inclusion being trivial, we have $(IR_c)^* = IR_c$. \square

Corollary 7.1.6. *If c is in the test ideal of a Stanley-Reisner ring $R = \frac{k[\mathbf{x}]}{P_1 \cap \dots \cap P_r}$, then R_c is F -regular.*

Proof. This is immediate from Corollary 7.1.4 and Lemma 7.1.5. \square

Now we use Theorem 5.4.2 to describe the test ideal of a Stanley-Reisner ring, giving a new proof of the following result due to Cowden [10, Theorem 3.6].

Theorem 7.1.7. *For a Stanley-Reisner ring $R = \frac{k[\mathbf{x}]}{\mathcal{J}}$ with minimal primes P_1, \dots, P_r , the test ideal of R is $\sum_{i=1}^r P_1 \cap \dots \cap \hat{P}_i \cap \dots \cap P_r$.*

Proof. Take $z \in I^*$. By Lemma 7.1.3, $z \in I + P_i$ for all minimal primes P_i of J . If $c \in P_1 \cap \dots \cap \hat{P}_i \cap \dots \cap P_r$ then $cz \in I + cP_i = I$. So $\sum_{i=1}^r P_1 \cap \dots \cap \hat{P}_i \cap \dots \cap P_r$ is contained in the test ideal.

To prove the converse, first note that the test ideal of R is a D-ideal (see [47, Theorem 2.2]). Thus the test ideal is a radical monomial ideal (Lemma 5.4.2). It suffices to show that each monomial generator of the test ideal is in $\sum_{i=1}^r P_1 \cap \dots \cap \hat{P}_i \cap \dots \cap P_r$. Let c be such a monomial. Without loss of generality, $c \notin \cup_{i=1}^t P_i$ and $c \in \cap_{i=t+1}^r P_i$. We may assume that $c = x_1 \cdots x_d$.

Now $R_c = \frac{k[\mathbf{x}]_c}{P_1 \cap \dots \cap P_t} = \frac{k[x_1, \dots, x_N]}{P_1 \cap \dots \cap P_t}[\frac{1}{x_1}, \dots, \frac{1}{x_d}]$ is F-regular (Corollary 7.1.6). The ring $T = \frac{k[x_1, \dots, x_N]}{P_1 \cap \dots \cap P_t}$ is a direct summand of R_c (it is the k -linear span of homogeneous elements which have degree 0 in the (inverted) variables, x_1, \dots, x_d). As a direct summand of an F-regular ring, T is also F-regular ([16, Proposition 4.12]). Then T is a normal ring ([16, Corollary 5.11]). Thus, T is a product of graded normal domains, $T = R_1 \times \dots \times R_s$. It follows that T has at least s homogeneous maximal ideals. But $T = \frac{k[x_1, \dots, x_N]}{P_1 \cap \dots \cap P_t}$ has a unique homogeneous maximal ideal. So T is a domain. From this it follows that $t = 1$; that is, $c \in P_2 \cap \dots \cap P_r$. \square

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