

Tight Closure and Differential Simplicity

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The behavior of the Hasse–Schmidt algebra under étale extension is used to show that the Hasse–Schmidt algebra of a smooth algebra of finite type over a field equals the ring of differential operators. These techniques show that the formation of Hasse–Schmidt derivations does not commute with localization, providing a counterexample to a question of Brown and Kuan; their conjecture is reformulated in terms of the Hasse–Schmidt algebra. These techniques also imply that a smooth domain R is differentially simple. Tight closure is used to show that the test ideal is Hasse–Schmidt stable. Indeed, differentially simple rings of prime characteristic are strongly F-regular.

Key Words: tight closure; differential simplicity; Hasse–Schmidt derivations; differential operators.

INTRODUCTION

The Hasse–Schmidt derivations (higher derivations) are a natural generalization of derivations. Given a k -algebra R , a Hasse–Schmidt derivation is a collection of k -linear maps $\{d_n: R \rightarrow R\}_{n \geq 0}$ such that $d_0 = id_R$ and

$$d_n(ab) = \sum_{i+j=n} d_i(a)d_j(b).$$

In particular, the first component d_1 is a k -derivation of R . The Hasse–Schmidt algebra $\text{HS}(R/k)$ is the R -algebra generated by all components d_n of all Hasse–Schmidt derivations. The Hasse–Schmidt algebra is a much studied generalization of the derivation algebra: in characteristic zero, each derivation d extends to a Hasse–Schmidt derivation $\{d_n = \frac{1}{n!}d^n\}$ and $\text{HS}(R/k)$ is just the algebra generated by derivations [14]. In prime characteristic the Hasse–Schmidt algebra retains many properties that are not

preserved for the derivation algebra; for example, minimal primes of R are Hasse–Schmidt stable, though not always stable under the k -derivations (see Traves [21, Chap. 4]).

Though the Hasse–Schmidt algebra was introduced in order to study number-theoretic properties of fields, it also has interesting geometric and ring-theoretic properties. For instance, we show that when R is a smooth algebra of finite type over a field k , then the Hasse–Schmidt algebra $\text{HS}(R/k)$ is precisely the ring of differential operators $D(R/k)$. This extends an observation of Grothendieck [6, 16.11.2]: The ring of differential operators on a smooth algebra of characteristic zero is generated by derivations. The proof relies on a variant of Noether normalization to factor the structure map $\text{Spec}(R) \rightarrow \text{Spec}(k)$ through an étale morphism. The behavior of various rings of operators under étale extensions is crucial. Másson’s observation [12] that differential operators behave well under étale extensions plays an important role. In order to understand the behavior of Hasse–Schmidt derivations on étale extensions, the Hasse–Schmidt derivations are reinterpreted as certain ring maps. This leads to a simple counterexample to an old question posed by Brown and Kuan [5]: Do Hasse–Schmidt derivations localize? Their conjecture is reformulated in terms of the Hasse–Schmidt algebra.

As an immediate consequence of our investigations, we see that every smooth k -algebra is a product of finitely many differentially simple domains (that is, domains R that are $\text{HS}(R)$ -simple). Much attention has been devoted to the study of differentially simple rings (see [4] and [10]). In studying equisingularity, Zariski showed that differentially simple rings are analytic products in characteristic zero [22]. Brown and Kuan [5] and Ishibashi [9] partially extended this result to prime characteristic by focusing on integrable derivations (derivations that are also elements of the Hasse–Schmidt algebra $\text{HS}(R/k)$).

In the second part of this paper (which can be read independent of the first) we determine a powerful constraint on the structure of differentially simple rings in prime characteristic: under very mild conditions, reduced differentially simple rings are strongly F -regular. This can be paraphrased by saying that under mild hypotheses differentially simple rings in prime characteristic determine log-terminal singularities.

This result is reminiscent of Smith’s characterization of strongly F -regular rings [19]: An F -finite ring R is strongly F -regular if and only if R is F -split and R is a simple $D(R)$ -module. Indeed, both results involve the theory of tight closure. The techniques developed to establish the main result are also applied to show that the test ideal is Hasse–Schmidt stable. In turn, this is used to glean information about the Hasse–Schmidt algebra itself.

1. HS (R/K) FOR R SMOOTH OVER K 1.1. *The Hasse–Schmidt Algebra*

The main aim of this paper is to study the Hasse–Schmidt algebra. Let R be an algebra of finite type over a field k . A *Hasse–Schmidt derivation* $\Delta = \{\delta_n\}_{n=0}^\infty$ is a sequence of k -linear endomorphisms of R such that $\delta_0 = id_R$ and

$$\delta_n(ab) = \sum_{i+j=n} \delta_i(a)\delta_j(b).$$

In the literature [3, 14], the Hasse–Schmidt derivations are sometimes called higher order operators. The *Hasse–Schmidt algebra* $\text{HS}(R/k)$ (or just $\text{HS}(R)$ when k is understood) is the subalgebra of $\text{End}_k(R)$ generated by all components δ_n of all Hasse–Schmidt derivations. The divided powers operators $\delta_n = \frac{1}{n!}(\partial^n/\partial x_i^n)$ on the polynomial ring $R = k[x_1, \dots, x_n]$ are the simplest example of a Hasse–Schmidt derivation.

The first component δ_1 of each Hasse–Schmidt derivation is itself a derivation of R . In characteristic zero, every derivation appears as the first component of a Hasse–Schmidt derivation and the Hasse–Schmidt algebra is generated by derivations (see Ribenboim [14]). For this reason, the Hasse–Schmidt algebra is often thought of as a characteristic-free analogue of the derivation algebra. The reader is cautioned that in prime characteristic the Hasse–Schmidt algebra need not be generated by derivations and not all derivations are *integrable*: they need not appear as the first component of a Hasse–Schmidt derivation. We present a simple example to emphasize this point; we will return to this example later to illustrate the difference between the Hasse–Schmidt algebra and the divided powers algebra on the derivations.

EXAMPLE 1.1. Consider the cone $R = k[x, y, z]/(xy - z^2)$, where k is a field of characteristic 2. The derivation $\frac{\partial}{\partial z}$ is not integrable. To see this, note that if $\{d_n\}$ is a Hasse–Schmidt derivation with $d_1 = \frac{\partial}{\partial z}$, then

$$d_2(z^2) = zd_2(z) + d_1(z)d_1(z) + d_2(z)z = (d_1(z))^2 = 1.$$

Now we produce a contradiction by showing that d_2 does not define an operator on R . We compute

$$\begin{aligned} 0 = d_2(0) &= d_2(xy - z^2) = d_2(xy) - d_2(z^2) \\ &= d_2(x)y + d_1(x)d_1(y) + xd_2(y) - (d_1(z))^2 \\ &= d_2(x)y + xd_2(y) - 1 \\ &\equiv -1 \pmod{(x, y, z)}, \end{aligned}$$

a contradiction. It follows that $\frac{\partial}{\partial z}$ is not integrable: no Hasse–Schmidt derivation $\{d_n\}$ has $d_1 = \frac{\partial}{\partial z}$.

The example relies on a behavior of Hasse–Schmidt derivations that is peculiar to prime characteristic. In order to explain this phenomenon, we interpret the Hasse–Schmidt derivations as ring endomorphisms.

Hasse–Schmidt derivations on R induce algebra endomorphisms of the power series algebra $R[[t]]$. Given a k -algebra R and a Hasse–Schmidt derivation $\Delta = \{\delta_n\} \subset \text{HS}(R/k)$, we form an algebra homomorphism

$$e^{t\Delta}: R[[t]] \rightarrow R[[t]]$$

as follows. First, extend the action of δ_n to $R[[t]]$ by linearity: $\delta_n(at^i) = \delta_n(a)t^i$. Then set

$$e^{t\Delta} = \delta_0 + \delta_1 t + \delta_2 t^2 + \dots$$

Now we check that $e^{t\Delta}$ is a ring map. It is k -linear by definition. To check that $e^{t\Delta}$ is multiplicative, it suffices to verify that $e^{t\Delta}(ab) = e^{t\Delta}(a)e^{t\Delta}(b)$ for a and b in R :

$$\begin{aligned} e^{t\Delta}(ab) &= \sum_{n=0}^{\infty} t^n \delta_n(ab) \\ &= \sum_{n=0}^{\infty} t^n \sum_{i+j=n} \delta_i(a)\delta_j(b) \\ &= \sum_{n=0}^{\infty} \sum_{i+j=n} t^i \delta_i(a)t^j \delta_j(b) \\ &= \left[\sum_{i=0}^{\infty} t^i \delta_i(a) \right] \left[\sum_{j=0}^{\infty} t^j \delta_j(b) \right] \\ &= e^{t\Delta}(a)e^{t\Delta}(b). \end{aligned}$$

Note that $e^{t\Delta}(t) = t$. This fact helps to characterize those endomorphisms ϕ of $R[[t]]$ that arise as maps of the form $e^{t\Delta}$.

THEOREM 1.1. *Let $R[[t]] \xrightarrow{\mu} R[[t]]/(t) \cong R$ be the quotient map. If ϕ is a k -algebra endomorphism of $R[[t]]$ such that $\phi(t) = t$ and $\mu \circ \phi$ induces the identity map on R , then $\phi = e^{t\Delta}$ for some Hasse–Schmidt derivation Δ .*

Proof. Suppose ϕ is a k -algebra endomorphism of $R[[t]]$ such that $\phi(t) = t$ and $(\mu \circ \phi)|_R = id_R$. For $a \in R$, write $\phi(a) = \sum_i d_i(a)t^i$. This defines k -linear maps $d_i: R \rightarrow R$. The map d_0 is the identity map: $d_0(a) = (\mu \circ \phi)|_R(a) = a$. Because ϕ is a ring map,

$$\sum_{i=0}^{\infty} d_i(ab)t^i = \phi(ab) = \phi(a)\phi(b) = \sum_{i=0}^{\infty} \left(\sum_{j+k=i} d_j(a)d_k(b) \right) t^i,$$

for $a, b \in R$. So $d_i(ab) = \sum_{j+k=i} d_j(a)d_k(b)$ and the maps $\Delta = \{d_i\}$ form a Hasse–Schmidt derivation. ■

We use this interpretation of Hasse–Schmidt derivations as algebra endomorphisms to show that in prime characteristic p , the action of Hasse–Schmidt derivations on p^e th powers satisfies an interesting relation.

COROLLARY 1.1. *If the characteristic of R is p and $\Delta = \{\delta_n\}$ is a Hasse–Schmidt derivation, then for $g \in R$,*

$$\delta_n(g^{p^e}) = \begin{cases} 0 & \text{if } p^e \text{ does not divide } n, \\ (\delta_{n/p^e}(g))^{p^e} & \text{if } p^e \text{ divides } n. \end{cases}$$

Proof. This is an immediate consequence of the following computation in $R[[t]]$ (equate coefficients of t):

$$\begin{aligned} g^{p^e} + (\delta_1(g)t)^{p^e} + (\delta_2(g)t^2)^{p^e} + \cdots &= (g + \delta_1(g)t + \delta_2(g)t^2 + \cdots)^{p^e} \\ &= e^{t\Delta}(g) \cdots e^{t\Delta}(g) \quad (p^e \text{ copies}) \\ &= e^{t\Delta}(g^{p^e}) \\ &= g^{p^e} + \delta_1(g^{p^e})t + \delta_2(g^{p^e})t^2 + \cdots. \end{aligned}$$

■

The interpretation of Hasse–Schmidt derivations as algebra endomorphisms also provides an example of a ring in which the Hasse–Schmidt algebra differs from the algebra generated by divided powers on derivations.

EXAMPLE 1.2. We return to Example 1.1: $R = k[x, y, z]/(xy - z^2)$, where k is a field of characteristic 2. The k -linear derivations on R are generated by $d_1 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ and $d_2 = \frac{\partial}{\partial z}$. From this it follows that no operator of the form $\sum_{a,b} P_{a,b} \frac{1}{a!} d_1^a \frac{1}{b!} d_2^b$ with $P_{a,b} \in R$ can send y to x . On the other hand, consider the second component of the Hasse–Schmidt derivation associated to the k -algebra map $\theta: R[[t]] \rightarrow R[[t]]$ that sends

$$\begin{aligned} x &\longrightarrow x + xt + xt^2 + \cdots \\ y &\longrightarrow y + yt + xt^2 + \cdots \\ z &\longrightarrow z + xt \end{aligned}$$

(the reader can easily check that the power series $\theta(x)$ and $\theta(y)$ can be completed subject to the relation $\theta(x)\theta(y) = \theta(z)^2$). The second component of this Hasse–Schmidt derivation is represented by the operator

$$\begin{aligned} \gamma &= x \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + x^2 \frac{1}{2!} \frac{\partial^2}{\partial x^2} + y^2 \frac{1}{2!} \frac{\partial^2}{\partial y^2} + x^2 \frac{1}{2!} \frac{\partial^2}{\partial z^2} + xy \frac{\partial}{\partial x} \frac{\partial}{\partial y} \\ &\quad + x^2 \frac{\partial}{\partial x} \frac{\partial}{\partial z} + xy \frac{\partial}{\partial y} \frac{\partial}{\partial z}. \end{aligned}$$

Clearly, $\gamma \in \text{HS}(R)$ sends y to x and so $\text{HS}(R)$ is not generated by divided powers on the k -derivations of R .

As a final application of the interpretation of Hasse–Schmidt derivations as algebra endomorphisms, we show that the Euler operator of a graded ring is an integrable derivation.

THEOREM 1.2. *If R is a graded ring of finite type with generators x_i of weight w_i , then the Euler operator $\theta = \sum w_i x_i (\partial / \partial x_i)$ is an integrable derivation.*

Proof. The first component of the algebra endomorphism determined by

$$x_i \mapsto x_i \left(\frac{1}{1-t} \right)^{w_i} = x_i + w_i x_i t + \dots$$

is the Euler operator. ■

1.2. Differential Operators

We treat the ring of differential operators $D(R/k)$, following Grothendieck [6]. This is a filtered subalgebra of $\text{End}_k(R)$; the differential operators of order $\leq n$ are defined by

$$D(R/k)_n = \{ \theta \in \text{End}_k(R) : [[\theta, a_0], a_1] \dots, a_n] = 0 \\ \text{for all } a_0, \dots, a_n \in R \}.$$

The operators of order zero are multiplication maps (represented by elements of R) and the k -algebra of operators of order less than or equal to 1 is identified with $R \oplus \text{Der}_k(R)$.

THEOREM 1.3. *The Hasse–Schmidt algebra $\text{HS}(R/k)$ is a subring of the ring of differential operators $D(R/k)$. Moreover, if $\Delta = \{ \delta_n \}$ is a Hasse–Schmidt derivation, then δ_n is a differential operator of order $\leq n$.*

Proof. We proceed by induction on n . The case $n = 0$ is clear. Suppose that $\delta_i \in D(R/k)_i$ for $0 \leq i \leq n - 1$. For a and b in R ,

$$\begin{aligned} [\delta_n, a](b) &= \delta_n(ab) - a\delta_n(b) \\ &= \sum_{i=1}^n \delta_i(a)\delta_{n-i}(b) \\ &= \left(\sum_{i=1}^n \delta_i(a)\delta_{n-i} \right)(b). \end{aligned}$$

By the induction hypothesis, the operator $[\delta_n, a] = \sum_{i=1}^n \delta_i(a)\delta_{n-i}$ is in $D(R/k)_{n-1}$ for each $a \in R$. This shows that $\delta_n \in D(R/k)_n$ and completes the induction. ■

We introduce a second way to describe the differential operators of order $\leq n$. First note that $R \otimes_k R$ acts on $\text{End}_k(R)$: $a \otimes b$ acts on the map θ to give the map $a \circ \theta \circ b$. The kernel J of the multiplication map $R \otimes_k R \rightarrow R$ is generated by elements of the form $1 \otimes a - a \otimes 1$ ($a \in R$) and

$$(1 \otimes a - a \otimes 1)\theta = [\theta, a].$$

Now it is easy to see that $\theta \in \text{End}_k(R)$ is a differential operator of order $\leq n$ if and only if $J^{n+1}\theta = 0$. These remarks lead to a canonical isomorphism,

$$D(R/k)_n = \text{Hom}_{R \otimes_k R}(P_{R/k}^n, \text{End}_k(R)) \cong \text{Hom}_R(P_{R/k}^n, R),$$

where $P_{R/k}^n = (R \otimes_k R)/J^{n+1}$ is the *module of n -jets*. Together with the map

$$\begin{aligned} d_{R/k}^n : R &\rightarrow R \otimes_k R \\ a &\mapsto 1 \otimes a \end{aligned}$$

the module $P_{R/k}^n$ acts as a universal object for the differential operators of order $\leq n$. That is, each differential operator δ of order $\leq n$ factors through $d_{R/k}^n$: there exists a unique R -module homomorphism $\theta: P_{R/k}^n \rightarrow R$ such that $\delta = \theta \circ d_n$.

EXAMPLE 1.3. Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a field k . Then

$$P_{R/k}^t = \frac{k[x_1, \dots, x_n, dx_1, \dots, dx_n]}{(x_1 - dx_1, \dots, x_n - dx_n)^{t+1}},$$

where x_i corresponds to $x_i \otimes 1$ and dx_i corresponds to $1 \otimes x_i$. Using this, it is easy to check that $P_{R/k}^t$ is the free R -module generated by the monomials of degree $\leq t$ in dx_1, \dots, dx_n . Then the differential operators of order $\leq t$ are generated by the divided power operators $(1/a_1!)(\partial^{a_1}/\partial x_1^{a_1}) \cdots (1/a_n!)(\partial^{a_n}/\partial x_n^{a_n})$ where $a_1 + \cdots + a_n \leq t$. Since each of these operators is in $\text{HS}(R)$ and $\text{HS}(R) \subseteq D(R)$, we see that $\text{HS}(k[x_1, \dots, x_n]) = D(k[x_1, \dots, x_n])$. Indeed, both algebras are equal to the algebra generated by divided powers on the derivations of R .

1.3. Extension of Operators to Étale Covers

Másson [12] used the n -jets to show that differential operators behave well under formally étale extensions. An extension $A \rightarrow B$ is a *formally étale extension* when for all commutative diagrams

$$\begin{array}{ccc} B & \xrightarrow{u} & C/J \\ f \uparrow & & \uparrow \\ A & \xrightarrow{v} & C \end{array}$$

(here C is a commutative ring, $J \subset C$ is a nilpotent ideal, and u, v are ring homomorphisms) there is a unique ring homomorphism $u': B \rightarrow C$ lifting u ; that is, u' uniquely completes the diagram

$$\begin{array}{ccc}
 B & \xrightarrow{u} & C/J \\
 \uparrow f & \dashrightarrow u' & \uparrow \\
 A & \xrightarrow{v} & C
 \end{array}$$

Remark 1.1. The reader is warned that there are several different conventions in use in the literature regarding the term formally étale. Matsumura [13] speaks about formally étale maps with reference to topologies on the rings. Our definition of formally étale is equivalent to his definition of étale, a formally étale map with respect to the discrete topology. Our notion of formally étale is sometimes referred to as étale of finite type.

THEOREM 1.4 (Másson [12]). *If $A \rightarrow B$ is a formally étale extension and A and B are k -algebras of finite type, then $D(B/k)_n \cong D(A/k)_n \otimes_A B$ and $D(B/k) \cong D(A/k) \otimes_A B$.*

Remark 1.2. The proof uses the fact that $P_{R/k}^n$ is an R -algebra and $d_{R/k}^n$ is a ring map. First one uses the universal property characterizing the n -jets to show that $P_{B/k}^n \cong P_{A/k}^n \otimes_A B$ and then the result follows from duality.

Since each component of a Hasse–Schmidt derivation is a differential operator, each component extends over formally étale extensions. The next result shows that these extensions patch together to give a Hasse–Schmidt derivation on the étale cover.

THEOREM 1.5. *Let $A \xrightarrow{f} B$ be a formally étale extension and suppose that $\Delta = \{\delta_n\} \subset \text{HS}(A/k)$ is a Hasse–Schmidt derivation on A . Then there exists a unique Hasse–Schmidt derivation $\Gamma = \{\gamma_n\}$ on B such that $f \circ \delta_n = \gamma_n \circ f$.*

Proof. Fix a non-negative integer m . Let $B[[t]]/(t^{m+1}) \xrightarrow{g} B$ be the quotient by $tB[[t]]$. Let $A \xrightarrow{f} A[[t]]/(t^{m+1})$ be the natural inclusion, $\iota(a) = a$, and note that $A \xrightarrow{f} B$ extends to a map $A[[t]]/(t^{m+1}) \xrightarrow{\hat{f}} B[[t]]/(t^{m+1})$ with $\hat{f}(t) = t$. Since $e^{t\Delta}$ preserves t -adic order, $e^{t\Delta}: A[[t]] \rightarrow A[[t]]$ induces an endomorphism $e_m^{t\Delta}$ of $A[[t]]/(t^{m+1})$. Then we have the map $\hat{f} \circ e_m^{t\Delta} \circ \iota$ and the following diagram.

$$\begin{array}{ccc}
 B & \xrightarrow{id_B} & B \\
 \uparrow f & \dashrightarrow e_m^\Gamma & \uparrow g \\
 A & \xrightarrow{\hat{f} \circ e_m^{t\Delta} \circ \iota} & \frac{B[[t]]}{(t^{m+1})}
 \end{array}$$

Here, the ring map e_m^{Γ} exists (and is unique) because B is étale of finite type over A . Forcing e_m^{Γ} to commute with multiplication by t , we extend e_m^{Γ} to a ring map $e_m^{\Gamma}: B[[t]]/(t^{m+1}) \rightarrow B[[t]]/(t^{m+1})$. Because B is étale of finite type over A , the maps e_m^{Γ} patch together (form a direct system) to give $e^{\Gamma}: B[[t]] \rightarrow B[[t]]$. Furthermore,

$$e^{\Gamma}(t) = e^{\Gamma}(1)t = e^{\Delta}(1)t = t$$

and $(g \circ e^{\Gamma})|_B = id_B$. By Theorem 1.1, e^{Γ} gives rise to a Hasse–Schmidt derivation $\Gamma = \{\gamma_n\}$ extending Δ . ■

This extends a result observed by both Brown [2] and Ribenboim [15]: Hasse–Schmidt derivations extend to localizations.

COROLLARY 1.2. *If $\Delta = \{\delta_n\}$ is a Hasse–Schmidt derivation on A over k and T is a multiplicative subset of A , then there is a unique Hasse–Schmidt derivation on $T^{-1}A$ extending Δ .*

Proof. This follows immediately from Theorem 1.5 because the localization map $A \rightarrow T^{-1}A$ is formally étale. ■

Brown and Kuan [5, p. 405] raise the question of whether every Hasse–Schmidt derivation on a localization $T^{-1}A$ is extended from A . This is not the case, as the following example shows.

EXAMPLE 1.4. Hasse–Schmidt derivations on $k[x, y, \frac{1}{y}]$ correspond to ring endomorphisms of $k[x, y, \frac{1}{y}][[t]]$ that send t to t and project to the identity on $k[x, y, \frac{1}{y}]$. Such an endomorphism θ is completely determined by the image of x and y . Let

$$\begin{aligned}\theta(x) &= x + d_1(x)t + d_2(x)t^2 + \cdots \\ \theta(y) &= y + d_1(y)t + d_2(y)t^2 + \cdots.\end{aligned}$$

Because x and y are algebraically independent in $k[x, y, \frac{1}{y}]$, we are free to choose any values for $d_i(x), d_j(y)$ in $k[x, y, \frac{1}{y}]$. Set $d_n(y) = 0$ and $d_n(x) = y^{-n}$. The map θ is just e^{Δ} where $\Delta = \{d_n\}$ is a Hasse–Schmidt derivation on $k[x, y, \frac{1}{y}]$. Such a Hasse–Schmidt derivation is the extension of a Hasse–Schmidt derivation $\Gamma = \{\gamma_n\}$ on $k[x, y]$ if there exists an element $g \in k[x, y]$ such that $\Gamma = g\Delta = \{g^n d_n\}$. Since the powers of y required to rationalize d_n grow exponentially, no such element g exists.

Remark 1.3. Brown and Kuan’s conjecture is best reformulated in terms of the Hasse–Schmidt algebra: If T is a multiplicatively closed set in R , then $\text{HS}(T^{-1}R) \cong \text{HS}(R) \otimes_R T^{-1}R$. Of course, this is true in characteristic zero (where $\text{HS}(R) = \text{der}(R)$). As well, when R is smooth over k , this is an immediate corollary of the next theorem (and Theorem 1.4). The case where R is singular and of characteristic p remains open.

1.4. Smooth Algebras

We investigate the nature of the Hasse–Schmidt algebra on a smooth k -algebra.

THEOREM 1.6. *Let R be a smooth algebra of finite type over a field k . Then the ring of differential operators equals the Hasse–Schmidt algebra: $D(R/k) = \text{HS}(R/k)$.*

Proof. Let m be a maximal ideal of R . Because R is smooth over k , there exists a regular system of parameters $x_1, \dots, x_n \in R_m$ giving rise to the diagram (see Grothendieck [6, 17.15.9])

$$\begin{array}{ccc}
 & A = k[x_1, \dots, x_n] & \\
 & \nearrow & \downarrow F \\
 k & \longrightarrow & R_m,
 \end{array}$$

where the injection F is formally étale and the ring A is a polynomial ring with coefficients in k . From Theorem 1.4,

$$D(R_m/k) \cong D(A/k) \otimes_A R_m. \tag{*}$$

In Example 1.3 we saw that

$$D(A/k) = \text{HS}(A/k).$$

By Theorem 1.5, we can lift each Hasse–Schmidt derivation on A to a Hasse–Schmidt derivation on R_m . Because the Hasse–Schmidt derivations on A generate $D(A/k)$, their liftings over F generate the R_m -algebra $D(R_m)$ (using $(*)$). So the algebra generated by the Hasse–Schmidt derivations on A lifts to a subalgebra H of $D(R_m)$ that equals $D(R_m)$. Since $H \subseteq \text{HS}(R_m) \subseteq D(R_m)$, $\text{HS}(R_m) = D(R_m)$ for all maximal ideals m of R . Using Corollary 1.2 and the same argument, each Hasse–Schmidt derivation of R extends to a Hasse–Schmidt derivation on R_m and

$$\text{HS}(R) \otimes_R R_m = \text{HS}(R_m) = D(R_m).$$

Now consider the R -module $\text{HS}_n(R/k) = D(R/k)_n \cap \text{HS}(R/k)$. Because $P_{R/k}^n$ is finitely presented, $D(R/k)_n = \text{Hom}_R(P_{R/k}^n, R)$ is a finitely generated R -module. Now $D(R/k)_n/\text{HS}_n(R/k)$ is locally zero:

$$\frac{D(R/k)_n}{\text{HS}_n(R/k)} \otimes_R R_m \cong \frac{D(R/k)_n \otimes_R R_m}{\text{HS}_n(R/k) \otimes_R R_m} = \frac{D(R_m/k)_n}{\text{HS}_n(R_m/k)} = 0.$$

So $D(R/k)_n = \text{HS}_n(R/k)$. Because this holds for all n and $D(R/k) = \lim D(R/k)_n$, the ring of differential operators $D(R/k)$ equals its subalgebra $\text{HS}(R/k)$. ■

This theorem is a characteristic-free analogue of a characteristic zero result due to Grothendieck.

COROLLARY 1.3 (Grothendieck [6, 16.11.2]). *If R is a smooth k -algebra of finite type, where k is a field of characteristic zero, then $D(R/k)$ is generated by derivations.*

Proof. In characteristic zero, $\text{HS}(R/k) = \text{der}(R/k)$. The result now follows from Theorem 1.6. ■

Remark 1.4. From the proof of Theorem 1.6, we see that the theorem and its corollary also hold for k -algebras R that are the localization of a smooth k -algebra of finite type.

Remark 1.5. Theorem 1.6 and its corollary also hold for divided powers on derivations: When R is smooth over k , the divided powers on the k -derivations generate $D(R)$. As we have seen in Example 1.2, these algebras need not agree when R is not smooth over k .

Now we show that if R is a smooth algebra over a field, then R is a finite product of D -simple domains.

THEOREM 1.7. *If R is a smooth algebra over a field k , then R is a finite product of D -simple domains. Similarly, R is also a finite product of differentially simple domains.*

Proof. The last statement follows immediately from the first, since $D(R) = \text{HS}(R)$ when R is a smooth k -algebra. Since R is smooth over a field, R is regular. Then R is a product of domains. If a product of rings is smooth, then each factor is smooth, so R is a product of smooth domains. Now, without loss of generality, we may suppose that R is a smooth domain.

Now we reduce to the local case. Suppose that R_m is $D(R_m/k)$ -simple for all maximal ideals m of R . If $I \subset R$ is a $D(R/k)$ -stable proper ideal, then there is a maximal ideal m containing I . Because R_m is a domain, IR_m is a nonzero $D(R_m/k)$ -stable proper ideal, a contradiction.

So suppose that (R, m) is a localization of a smooth k -algebra. Let x_1, \dots, x_d be a regular system of parameters for m . Suppose that I is an ideal of R and that $r \in I$ is nonzero. Then there is some integer N such that $r \in m^N \setminus m^{N+1}$. Using multi-index notation, write

$$\left[r - \sum_{|s|=N} c_s x^s \right] \in m^{N+1},$$

where the c_s are units (or zero) and one of the c_s is nonzero, say c_e . By Theorem 1.6, there is a differential operator D , such that D has order $\leq N$ and

$$D(x^s) = \binom{s}{e} x^{s-e}.$$

The operator D is the extension of the operator

$$\frac{1}{e_1!} \frac{\partial^{e_1}}{\partial x_1^{e_1}} \cdots \frac{1}{e_n!} \frac{\partial^{e_n}}{\partial x_n^{e_n}}$$

on the intermediate algebra $A = k[\mathbf{x}] = k[x_1, \dots, x_n]$ appearing in Theorem 1.6. Let $t = c_e^{-1}r \in I$. Then $D(t) = 1 + D(m^{N+1})$. Since D has order $\leq N$, $D(m^{N+1}) \subseteq m$ and so $D(t)$ is a unit in the local ring R . Thus $I = R$. ■

2. DIFFERENTIAL SIMPLICITY AND TIGHT CLOSURE

Having shown that smooth domains are differentially simple, we investigate the algebraic properties of differentially simple rings. Zariski [22] showed that differentially simple domains are analytic products in characteristic zero. We restrict our attention to the case of prime characteristic. Here the techniques of tight closure are used to show that under mild conditions, the test ideal is Hasse–Schmidt stable. Indeed, reduced differentially simple rings of finite type over a perfect field are strongly F -regular. As an application, we use these results to investigate the structure of the Hasse–Schmidt algebra of certain graded rings. This section is largely independent of Section 1.

2.1. Hasse–Schmidt and Frobenius

Let R be a local ring, essentially of finite type, with coefficient field k and characteristic p . We remind the reader that a *Hasse–Schmidt derivation* $\{\delta_n\}_{n=0}^\infty$ is a collection of k -linear endomorphisms of the ring R such that $\delta_0 = id_R$ and

$$\delta_n(rs) = \sum_{i+j=n} \delta_i(r)\delta_j(s)$$

for r and s arbitrary elements of R . The *Hasse–Schmidt algebra* $HS(R)$ is the R -subalgebra of the endomorphism ring generated by all the components δ_n of all the Hasse–Schmidt derivations on R . An R -module M is a *Hasse–Schmidt module* if it admits an action of $HS(R)$ consistent with the action of $HS(R)$ on R ,

$$\delta_n(rm) = \sum_{i+j=n} \delta_i(r)\delta_j(m),$$

for all $r \in R$ and $m \in M$. Since the Hasse–Schmidt algebra is a prime-characteristic analogue of the derivation algebra, we say that a simple Hasse–Schmidt module is *differentially simple*.

Now we define the action of $\text{HS}(R)$ on the Frobenius power $M^{[p^e]}$ of a Hasse–Schmidt module M . The e th Frobenius power of R , $F^e(R)$, is an R -bimodule isomorphic (as a left R -module) to R , but admitting a twisted right action: for $a \in F^e(R)$ and $r \in R$,

$$a \cdot r = ar^{p^e}.$$

If M is an R -module, then $F^e(M) = F^e(R) \otimes_R M$. Then, for $s \in R$, $r \in F^e(R)$, and $m \in M$, we have

$$\begin{aligned} s(r \otimes m) &= sr \otimes m \\ \text{and } r \otimes sm &= rs^{p^e} \otimes m. \end{aligned}$$

Define

$$\delta_n(r \otimes m) = \sum_{i+jp^e=n} \delta_i(r) \otimes \delta_j(m).$$

We first prove that this constitutes a well-defined action. We must show that $\delta_n(c \otimes rm) = \delta_n(cr^{p^e} \otimes m)$:

$$\begin{aligned} \delta_n(c \otimes rm) &= \sum_{i+jp^e=n} \delta_i(c) \otimes \delta_j(rm) \\ &= \sum_{i+jp^e=n} \delta_i(c) \otimes \sum_{k+l=j} \delta_k(r) \delta_l(m) \\ &= \sum_{i+(k+l)p^e=n} \delta_i(c) (\delta_k(r))^{p^e} \otimes \delta_l(m) \\ &= \sum_{i+(k+l)p^e=n} \delta_i(c) \delta_{kp^e}(r^{p^e}) \otimes \delta_l(m) \quad (*) \\ &= \sum_{s+lp^e=n} \delta_s(cr^{p^e}) \otimes \delta_l(m) \quad (*) \\ &= \delta_n(cr^{p^e} \otimes m). \end{aligned}$$

The next-to-last step involves the substitution $s = i + kp^e$ and the steps labeled (*) use Corollary 1.1. This computation shows that the action is well-defined.

The action is defined in this way in order to establish a strong linearity result: If $c \otimes m \in F^e(M)$ then

$$\delta_n(c \otimes m) = \delta_n(c) \otimes m$$

for $p^e > n$.

2.2. Tight Closure

The *tight closure* of an ideal I in a reduced Noetherian ring of characteristic p is a possibly larger ideal I^* . An element $x \in R$ is in the tight closure I^* if there exists an element c , not in any minimal prime of R , such that

$$cx^{p^e} \in I^{[p^e]}$$

for all $e \gg 0$. Here $I^{[p^e]}$ denotes the ideal generated by all p^e th powers of elements in I . In what follows the letter q is often used to indicate a power of p : $q = p^e$.

We can also define the tight closure of submodules of an R -module. If $N \subset M$ are R -modules, we say that $x \in M$ is in the tight closure N_M^* of N if there exists an element c , not in any minimal prime of R , such that

$$c \otimes x \in N^{[p^e]} = \text{Image}(F^e(N) \rightarrow F^e(M))$$

for all $e \gg 0$.

In the definition of tight closure the element c appears to depend on both x and I . Since R is Noetherian, the dependence on x is superficial; however, the dependence on I is a more delicate matter. We want to find elements c that work in all tight closure tests, not only for all ideals in R but also for all R -modules. To this end, call an element c a *q-weak test element* if $cx^{q'} \in I^{[q']}$ for all $q' > q$ whenever $x \in I^*$. A 1-weak test element is called a *test element*. Hochster and Huneke [7, Theorem 3.4] have shown that “geometric” rings possess test elements; in particular, reduced essentially finite algebras defined over a field admit test elements (in the paper they only claim the result for ideals, though Hochster pointed out that these elements (and their proof!) work for all modules as well).

The test elements generate an ideal $\tau(R)$, the test ideal; the weak test elements also generate an ideal, although we focus attention on a slightly different ideal, the weak test ideal of the module M ,

$$\tau_w(M) = \bigcap_{q \gg 0} \bigcap_{N \subset M} \left(N_M^{[q]} :_R (N_M^*)^{[q]} \right).$$

Here the first intersection is interpreted in the following way: If $\{J_\lambda\}$ is a family of ideals in a Noetherian ring indexed by an increasing sequence of integers, then $\bigcap_{\lambda \geq t} J_\lambda$ is an increasing sequence of ideals; these stabilize at the ideal $\bigcap_{\lambda \gg 0} J_\lambda$.

The *exponent* of the weak test ideal $\tau_w(M)$ is the smallest power $Q = p^e$ such that

$$\bigcap_{q \geq Q} \bigcap_{N \subset M} \left(N_M^{[q]} :_R (N_M^*)^{[q]} \right) = \tau_w(M).$$

When R is Noetherian, the exponent of the weak test ideal is finite. Smith [19] showed that when R is a finite R^p module, then $\tau_w(R)$ is stable under the action of the differential operators $D(R)$; in particular, $\tau_w(R)$ is HS(R)-stable. We extend this result to Hasse–Schmidt modules.

THEOREM 2.1. *If M is a Hasse–Schmidt module then $\tau_w(M)$ is a non-zero Hasse–Schmidt stable ideal.*

Proof. Let c be an element of $\tau_w(M)$, N be a submodule of M , and $\{\delta_n\}$ be a Hasse–Schmidt derivation. Take $x \in (N_M^*)^{[p^e]}$. Then $c \otimes x \in N_M^{[p^e]}$ for p^e greater than the exponent Q . For $p^e > \max(n, Q)$, we have:

$$\begin{aligned} \delta_n(c) \otimes x &= \delta_n(c \otimes x) \\ &= \delta_n\left(\sum r_i \otimes n_i\right) \quad (n_i \in N) \\ &= \sum \delta_n(r_i) \otimes n_i \in N_M^{[p^e]}. \end{aligned}$$

Thus, $\delta_n(c) \in \cap_N(N_M^{[q]} :_R (N_M^*)^{[q]})$ for all $q \gg 0$ and so $\delta_n(c) \in \tau_w(M)$. Since there exist test elements that work for all modules, $\tau_w(M) \neq 0$. ■

We quickly review some basic terminology from the theory of tight closure. When the maps $R^p \rightarrow R$ sending 1 to c split as maps of R^p -modules for all $c \in R \setminus \{0\}$, the ring is said to be *strongly F -regular* and when this happens for $c = 1$, we say that the ring is *F -split*. Strongly F -regular rings correspond to interesting singular varieties; for details, see Smith [20]. Smith showed that if (R, m) is a reduced local finite R^p -module such that $0_E^* = 0$ then R is strongly F -regular (here $E = E_R(R/m)$ is an injective hull of the residue field of R ; see [18, 7.1.2] or [11, 2.9]). Indeed, the support of the ideal $\text{Ann}_R(0_E^*)$ determines the non-strongly F -regular points in $\text{Spec}(R)$ (see Lyubeznick and Smith [11]). Our aim in the next section is to show that differentially simple rings are strongly F -regular.

The test ideal is also related to tight closure in E . If $N \subseteq M$ then the *finitistic tight closure* N_M^{*fg} of N is the limit of the tight closures of N in finitely generated submodules of M ,

$$N_M^{*fg} = \bigcup_H (N \cap H)_H^*,$$

where the union is taken over all finitely generated submodules $H \subset M$. In general, $N_M^{*fg} \subseteq N_M^*$. If R is a local Noetherian ring, then the test ideal is $\tau(R) = \text{Ann}_R(0_E^{*fg})$ (see Hochster and Huneke [8, Proposition 8.23]).

2.3. Hasse–Schmidt Stability and Tight Closure

In order to guarantee the existence of test elements, we introduce a *blanket hypothesis*: all rings treated in this section are reduced excellent algebras over a field of characteristic $p > 0$. Furthermore, we assume that R is *F -finite*: R is finite over R^p .

THEOREM 2.2. *Tight closure respects stability: if $N \subset M$ are Hasse–Schmidt stable modules, then N_M^* is also a Hasse–Schmidt stable module.*

Proof. Take $x \in N_M^*$ and c a weak test element for M . Then $c \otimes x \in N^{[q]}$ for all q larger than the exponent Q . Let $\{\delta_n\}$ be a Hasse–Schmidt derivation. In order to study the action of the component δ_n , consider the action of the component δ_{nq} for $q > Q$,

$$\begin{aligned} \delta_{nq}(c \otimes x) &= \sum_{i+jq=nq} \delta_i(c) \otimes \delta_j(x) \\ &\in \delta_{nq}(N^{[q]}) \subset (\delta_n(N) + \cdots + \delta_1(N) + N)^{[q]} \\ &= N^{[q]}, \end{aligned}$$

the last line following because N is Hasse–Schmidt stable. It follows that

$$\sum_{i+jq=nq} \delta_i(c) \otimes \delta_j(x) \in N^{[q]}. \quad (**)$$

We prove that $\delta_j(x) \in N_M^*$ by induction. The result is clearly true for $j = 0$. Suppose the result for $j < n$. By Theorem 2.1, each $\delta_i(c)$ is also a weak test element for M . Together with the induction hypothesis, this shows that $\delta_{(n-j)q}(c) \otimes \delta_j(x) \in N^{[q]}$ for $q > Q$ and $j < n$. Putting this together with (**) we get

$$c \otimes \delta_n(x) \in N^{[q]}$$

for $q > Q$. That is, $\delta_n(x) \in N_M^*$. Thus N_M^* is Hasse–Schmidt stable. ■

THEOREM 2.3. *Finitistic tight closure respects stability: if $N \subseteq M$ are Hasse–Schmidt modules, then N_M^{*fg} is also a Hasse–Schmidt stable module.*

Proof. The proof is similar to the proof of Theorem 2.2 but there are some subtle differences. Let $\{\delta_n\}$ be a Hasse–Schmidt derivation. Let c be a weak test element for R . Then c and each $\delta_j(c)$ act as weak test elements in all finitistic tight closure tests for modules. Take $x \in N_M^{*fg}$; that is, there is a finite submodule $H \subseteq M$ such that $x \in (N \cap H)_H^*$. Then $c \otimes x \in (N \cap H)^{[q]}$ for all q larger than the exponent Q of H . We need to show that $\delta_n(x) \in (N \cap \Gamma)_\Gamma^*$ for some finite submodule $\Gamma \subseteq M$. Now for $q > Q$,

$$\begin{aligned} \delta_{nq}(c \otimes x) &= \sum_{i+jq=nq} \delta_i(c) \otimes \delta_j(x) \\ &\in \delta_{nq}((N \cap H)^{[q]}) \subset (\delta_n(N \cap H) + \cdots + \delta_1(N \cap H) + N \cap H)^{[q]} \\ &\subset (\delta_n(N) \cap \delta_n(H) + \cdots + \delta_1(N) \cap \delta_1(H) + N \cap H)^{[q]} \\ &= (N \cap \delta_n(H) + \cdots + N \cap \delta_1(H) + N \cap H)^{[q]} \\ &= ((N \cap (\delta_n(H) + \cdots + \delta_1(H) + H))^{[q]}). \end{aligned}$$

Now $\Gamma = \delta_n(H) + \cdots + \delta_1(H) + H$ is a finite submodule of M . The rest of the proof of Theorem 2.2 carries through to show that $c \otimes \delta_n(x) \in (N \cap \Gamma)^{[q]}$ for all $q > Q$. That is, $\delta_n(x) \in (N \cap \Gamma)_\Gamma^*$ and $\delta_n(x) \in N_M^{*fg}$. ■

In order to apply Theorems 2.2 and 2.3, we indicate the manner in which the injective hull $E = E_R(R/m)$ of a local ring is a Hasse–Schmidt module. First the ring of differential operators $D(R)$ acts naturally on $D(R, R/m)$ by precomposition of operators. This induces an action of $\text{HS}(R)$ on $D(R, R/m)$ since $\text{HS}(R) \subset D(R)$. Now the Hasse–Schmidt action on E arises from the identification of E with $D(R, R/m)$:

LEMMA 2.1. $E = D(R, R/m)$.

Proof. This is known to follow from the identification of each object with the continuous homomorphisms from R to R/m :

$$\begin{aligned} E &= \lim E_{R/m^t}(R/m) \\ &= \lim \text{Hom}_k(R/m^t, R/m) \\ &= \lim D_{t-1}(R, R/m) \\ &= D(R, R/m). \end{aligned}$$

■

COROLLARY 2.1. 0_E^* and 0_E^{*fg} are Hasse–Schmidt stable submodules of E .

Proof. By Lemma 2.1, $0 \subset E$ are Hasse–Schmidt modules. Now the result follows from Theorems 2.2 and 2.3. ■

LEMMA 2.2. If N and P are Hasse–Schmidt modules then so is $(N :_R P)$.

Proof. Let $a \in R$ be an arbitrary element of $(N :_R P)$. We establish that $\delta_m(a) \in (N :_R P)$ by induction on m . Suppose that this is true for $m < M$, and note that for any $p \in P$,

$$\delta_M(ap) = \sum_{i+j=M} \delta_i(a)\delta_j(p) \equiv \delta_M(a)p \pmod{N}.$$

Since $ap \in N$, $\delta_M(ap) \in N$ and so $\delta_M(a)p \in N$. Thus $\delta_M(a) \in (N :_R P)$ and $(N :_R P)$ is $\text{HS}(R)$ -stable. ■

THEOREM 2.4. Let R be a reduced excellent local ring that is finite over R^p . Then the test ideal $\tau(R)$ is Hasse–Schmidt stable. Furthermore, if R is differentially simple then R is strongly F -regular.

Proof. By Corollary 2.1 and Lemma 2.2 we see that $\tau(R) = \text{Ann}_R(0_E^{*fg})$ is Hasse–Schmidt stable. We also see that the ideal $\text{Ann}_R(0_E^*) = (0 :_R 0_E^*)$ is Hasse–Schmidt stable. Noting that [7, Theorem 3.4] extends to modules, we see that there exist non-zero test elements for E ; so $\text{Ann}_R(0_E^*)$ is non-zero. Since R is differentially simple, we conclude that $\text{Ann}_R(0_E^*) = R$ and $0_E^* = 0$. Now the result follows from Smith’s characterization of strongly F -regular rings [18, 7.1.2]. ■

COROLLARY 2.2. *Let R be a reduced excellent local ring that is finite over R^p . If R is differentially simple, then R is F -split.*

The Hasse–Schmidt stability of the test ideal constrains the Hasse–Schmidt algebra of certain isolated singularities. We need some further results on Hasse–Schmidt stability to see this.

LEMMA 2.3. *The associated primes of $\text{HS}(R)$ -stable ideals are $\text{HS}(R)$ -stable.*

Proof. This is an easy extension of a result due to Seidenberg [17]. A complete proof can be found in Brown [2]. ■

COROLLARY 2.3. *The radical of a Hasse–Schmidt stable ideal is Hasse–Schmidt stable.*

Proof. The radical of an ideal is the intersection of its minimal primes. Each of these primes is $\text{HS}(R)$ -stable, thus so is their intersection. ■

LEMMA 2.4. *Products of Hasse–Schmidt stable ideals are Hasse–Schmidt stable. In particular, powers of Hasse–Schmidt stable ideals are Hasse–Schmidt stable.*

Proof. Suppose that I and J are $\text{HS}(R)$ -stable ideals and $\{\delta_n\}$ is a Hasse–Schmidt derivation on R . Let $i \in I$ and $j \in J$. Then

$$\delta_n(ij) = \sum_{t=0}^n \delta_t(i)\delta_{n-t}(j) \in IJ.$$

So products of Hasse–Schmidt stable ideals are Hasse–Schmidt stable. ■

Now we can constrain the behavior of Hasse–Schmidt derivations on certain isolated singularities.

THEOREM 2.5. *If (R, m) is an isolated non-strongly F -regular point, then all powers of m are Hasse–Schmidt stable. If R is graded and m is the maximal homogeneous ideal, then no component of a Hasse–Schmidt derivation can lower degree.*

Proof. The conditions on R force the test ideal $\tau(R)$ to be m -primary. Now $m = \sqrt{\tau(R)}$ is Hasse–Schmidt stable by Corollary 2.1 and Corollary 2.3. Powers of m are $\text{HS}(R)$ -stable by Lemma 2.4. ■

EXAMPLE 2.1. The cubic cone $R = k[x, y, z]/(x^3 + y^3 + z^3)$ ($\text{char}(k) \neq 3$) is not weakly F -regular (since $z^2 \in (x, y)^*$, but $z^2 \notin (x, y)$) so it is not strongly F -regular. Thus all powers of $m = (x, y, z)$ are Hasse–Schmidt stable and no operator in the Hasse–Schmidt algebra lowers degree in R . The question of whether every differential operator on R has nonnegative degree remains open in prime characteristic, although the result is well known in characteristic zero (see [1]).

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