Differential Operators and Invariant Theory

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Plan

- **Invariant Theory**
  - **Rings of invariants, R^G**
  - **Derksen's algorithm**

- **Differential symmetries of R^G**
  - **Ideals in the Weyl algebra**
  - **The symmetry algebra and $D(R^G)$**
  - **Computing $D(R^G)$**

Rings of Invariants

When a group G acts on X = Spec(R), it is customary to consider the categorical quotient X//G = Spec(R^G).

The ring R^G is the ring of invariants:

$$\boxed{R^G = \{ r \in R: \ g \bullet r = r \text{ for all } g \in G\}.}$$

<u>Ex</u>: If G = <σ: σ^2 = id> acts on R=C[x,y] via $\sigma \bullet x = -x$ and $\sigma \bullet y = -y$ then $R^G = C[x^2, xy, y^2]$.

Example: binary forms

The geometry of (pairs of) points on P^1 is controlled by the (deg 2) forms.

Let $X = \{ax^2 + bxy + cy^2\} = \mathrm{Proj}\ C[a,b,c]$

$G = SL_2C$ acts on P^1 and moves the roots of a form, so G acts on X:

$$\boxed{\begin{aligned} g\bullet a &= g_{1,1}^2\, a + 2g_{1,1}g_{2,1}\, b + g_{2,1}^2\, c \\ g\bullet b &= g_{1,1}g_{1,2}a + (g_{1,1}g_{2,2}+g_{1,2}g_{2,1})b + g_{2,1}g_{2,2}c \\ g\bullet c &= g_{1,2}^2\, a + 2g_{1,2}g_{2,2}\, b + g_{2,2}^2\, c \end{aligned}}$$

$R^G = C[a,b,c]^{SL_2C} = C[b^2\text{-}4ac]$

Properties of R^G

It is generally difficult to compute R^G explicitly (c.f. Takashi Wada's talk).

Gordan and Hilbert showed that R^G is finitely generated when G is lin. reductive and Nagata gave a counterexample when G is not lin. reductive.

Computing invariants

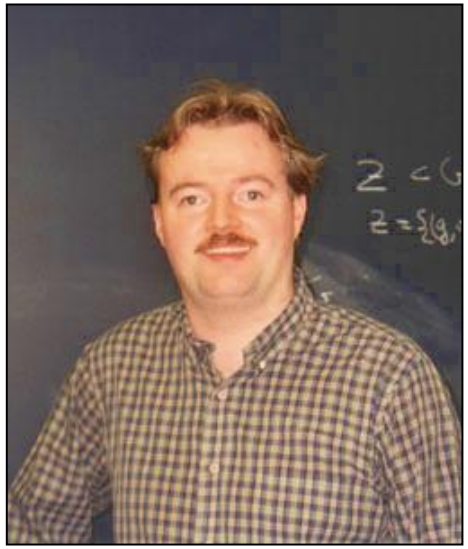
Several methods:

(1) Gordan's symbolic calculus (P. Olver)

(2) Cayley's omega process

(3) Lie algebra methods (Sturmfels)

(4) Derksen's algorithm



Harm Derksen

Gregor Kemper

Derksen's Algorithm

(1) Hilbert ideal I = ideal of R gen by $R^G_{>0}$

(2) To find I, we first look at the map

$$\psi : G \times X \to X \times X \qquad \mathrm{B} = \overline{\mathrm{im}(\psi)}$$

$$(\mathrm{g}, \mathrm{x}) \mapsto (\mathrm{x}, \mathrm{g} \bullet \mathrm{x}) \qquad \beta = \mathrm{ideal}(\mathrm{B})$$

$$\text{Hilbert - Mumford Criterion :}$$

$$\mathrm{B} \cap (\mathrm{X} \times \{0\}) = \mathrm{V}(I) \times \{0\}$$

$$\beta + (\mathrm{y}_1, \ldots, \mathrm{y}_\mathrm{n}) = I + (\mathrm{y}_1, \ldots, \mathrm{y}_\mathrm{n})$$

Compute the ideal β by elimination and set y's to 0 to get gens for the Hilbert ideal.

(3) The gens of I may not be invariants but we can average them over the group action to get invariants that generate I and R^G.

Example: binary forms

To describe the map $\psi : G \times X \to X \times X$

we need to represent $G = SL_2C$ as a variety :

$$G = V(g_{1,1}g_{2,2} - g_{2,1}g_{1,2} - 1) \subset C[g_{1,1}, g_{1,2}, g_{2,1}, g_{2,2}].$$

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Differential symmetries

One way to study any ring of functions is to understand the ring as the solutions to a system of differential equations.

Cayley developed a system of diff. eqs that characterize the invariants of forms (the $G=SL_2C$ case).

Arthur Cayley

Passage to GL_2C

The Lie algebra method works with GL_2C rather than SL_2C. There is a close connection between the invariants of these two groups.

A function f is said to be an "invariant of index γ" if $\boxed{g \bullet f = (\det g)^{\gamma} f}$ for all $g \in GL_2C$.

<u>Theorem</u>: For binary forms, GL_2C-invariants of index γ = homog. SL_2C-invariants of degree $2\gamma/d$.

Example: degree 2 forms

Degree 2 form: $F(x,y) = ax^2 + bxy + cy^2$.

$$g\bullet a = g_{1,1}^2\, a + 2g_{1,1}g_{2,1}\, b + g_{2,1}^2\, c$$
$$g\bullet b = g_{1,1}g_{1,2}a + (g_{1,1}g_{2,2}+g_{1,2}g_{2,1})b + g_{2,1}g_{2,2}c$$
$$g\bullet c = g_{1,2}^2\, a + 2g_{1,2}g_{2,2}\, b + g_{2,2}^2\, c$$

$$\text{If } g = \begin{bmatrix} \lambda & 0 \\ 0 & \tau \end{bmatrix} : g\bullet a = \lambda^2 a,\ \ g\bullet b = \lambda\tau\, b,\ \ g\bullet c = \tau^2 c$$

Now if $f(a,b,c)$ has index γ then
$f(\lambda^2 a, \lambda\tau b, \tau^2 c) = (\lambda\tau)^\gamma f(a,b,c)$. This forces

$$(2a\partial_a + 1b\partial_b + 0c\partial_c)f = \gamma f$$

$$(0a\partial_a + 1b\partial_b + 2c\partial_c)f = \gamma f$$

Example continued

$$(2a\partial_a + 1b\partial_b + 0c\partial_c)f = \gamma f$$

$$(0a\partial_a + 1b\partial_b + 2c\partial_c)f = \gamma f$$

There are two other types of matrices that gen. GL_2C:

$\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$ gives the condition $(1a\partial_b + 2b\partial_c)f = 0$ and

$\begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix}$ gives the condition $(2b\partial_a + 1c\partial_b)f = 0.$

Cayley's Differential System

$$(2a\partial_a + 1b\partial_b + 0c\partial_c)f = \gamma f$$

$$(0a\partial_a + 1b\partial_b + 2c\partial_c)f = \gamma f$$

$$(1a\partial_b + 2b\partial_c)f = 0$$

$$(2b\partial_a + 1c\partial_b)f = 0$$

$$\mathbf{ax^2 + bxy + cy^2} \longrightarrow \mathbf{a_2x^2 + a_1xy + a_0y^2}$$

$$\longrightarrow \sum a_i x^i y^{d-i}$$

Cayley's Differential System

$$\left(\sum i a_i \partial_i - \gamma\right) f = 0$$

$$\left(\sum (d-2i) a_i \partial_i\right) f = 0$$

$$\left(\sum (d-i) a_{i+1} \partial_i\right) f = 0$$

$$\left(\sum i a_{i-1} \partial_i\right) f = 0$$

$$ax^2 + bxy + cy^2 \longrightarrow a_2x^2 + a_1xy + a_0y^2$$

$$\longrightarrow \sum a_i x^i y^{d-i}$$

Ideals in the Weyl algebra

We can make Cayley's system into an algebraic object using the Weyl algebra.

Weyl algebra: $W = C[x_1,\ldots,x_n]\langle\partial_1,\ldots,\partial_n\rangle$

Systems of linear PDEs $\Longleftrightarrow$ left ideals in W

• f satisfies $\Delta_1 f = \Delta_2 f = \ldots = 0$ $\Longleftrightarrow$ f is annihilated by the left ideal $W(\Delta_1,\Delta_2,\ldots)$

Solutions to system of DEs correspond to elements of $\mathrm{Hom}_W(W/J, C[x_1,\ldots,x_n])$

Cayley's Differential System

$$\left(\sum i a_i \partial_i - \gamma\right) f = 0$$

$$\left(\sum (d-2i) a_i \partial_i\right) f = 0$$

$$\left(\sum (d-i) a_{i+1} \partial_i\right) f = 0$$

$$\left(\sum i a_{i-1} \partial_i\right) f = 0$$

J_d : characterizes invariants

$J_d(\gamma)$: invariants of order γ

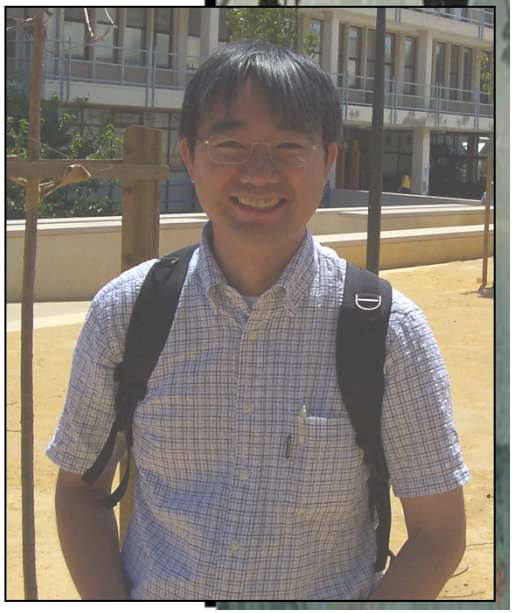
Symmetry algebra

$$S(W/J) = \frac{\{\Delta \in W : J\Delta \subseteq J\}}{J}$$

M. Saito and I used the symmetry algebra to study systems of hypergeometric DEs.

A: matrix, β: complex vector $\Rightarrow$ $H_A(\beta)$
$\Delta \in S(W/J_A)$

Then f is a soln to $H_A(\beta)$ $\Rightarrow$
$\Delta \bullet f$ is a soln to $H_A(\beta')$ for some β'.



Symmetry algebra

$$\boxed{S(W/J) = \frac{\{\Delta \in W : J\Delta \subseteq J\}}{J}}$$

$$\underline{\text{Theorem}}: \; S(W/J_d) = \frac{\{\Delta \in W : \Delta \bullet R^G \subseteq R^G\}}{\{\Delta \in W : \Delta \bullet R^G = 0\}} \subseteq D(R^G)$$

Equality holds when $J_d = \mathrm{Ann}_W(R^G)$.

<u>Question</u>: How can we compute $S(W/J_d)$?

This looks hard so we pass to the study of $D(R^G)$.

Differential Operators

If R is a k-algebra then $D(R) \subset \mathrm{End}_k R$.

In general D(R) is very difficult to compute and may not be finitely generated.

One nice example: $D(C[x_1,..,x_n]) = W$.

G acts on $C[x_1,\ldots,x_n]$ via the matrix M and this induces an action on the ∂_i's via the matrix $(M^{-1})^T$.

So it makes sense to ask for $D(R)^G = W^G$, the ring of invariant differential operators.

$D(R)^G$ vs. $D(R^G)$

Unfortunately, $D(R)^G$ is not equal to $D(R^G)$.

π: $R^G \to R$ induces π^*: $D(R)^G \to D(R^G)$ by restriction

$$\begin{tikzcd}
R \arrow[r, "\theta\in D(R)^G"] & R \arrow[d, "\text{Reynolds}"] \\
R^G \arrow[u] \arrow[r, dashed, "\pi^*\theta\in D(R^G)"'] & R^G
\end{tikzcd}$$

<u>Theorem</u>: if $J_d = \mathrm{Ann}_W R^G$ then

$$\mathrm{im}(\pi^*) \subset S(W/J_d) \subset D(R^G)$$

Failure of surjectivity

We've got a map π^*: $D(R)^G \to D(R^G)$.

Have $\mathrm{Im}(\pi^*) \subset S(D(R)/J) \subset D(R^G)$.

Musson and Van den Bergh showed that the map π^* may not be surjective (G = torus).

Ian Musson

M. Van den Bergh

Surjective when it counts

Schwarz showed that the map π^* is surjective in many cases of interest. In fact, he showed that the Levasseur-Stafford Alternative holds for SL_2C representations:

Either (1) R^G is regular or
(2) the map π^* is graded surjective

In most cases R^G is not regular and so π^* is (graded) surjective. Then: $\mathrm{im}(\pi^*) = S(W/J_d) = D(R^G)$.

Gerry Schwarz

Graded Surjectivity

The Weyl algebra D(R)=W is filtered by order (degree in the ∂'s). The associated graded ring is just a polynomial ring in 2n variables.

Saying that the map π^*: $D(R)^G \rightarrow D(R^G)$ is graded surjective means that the induced map $[grD(R)^G] \rightarrow grD(R^G)$ is surjective.

So to produce generators of $grD(R^G)$ (and hence $D(R^G)$) it is enough to get generators of $grD(R)^G$.

Computing $D(R^G)$

Since $grD(R)^G = [grD(R)]^G$ is a poly. ring, we can use Derksen's algorithm to compute $[grD(R)]^G$. These generators lift immediately to generators of $D(R^G)$.

Ex. $D(C[a_0,a_1,a_2]^{SL_2C}) =$
$C[a_1^2-4a_0a_2,\ a_0\partial_0+a_1\partial_1+a_2\partial_2,\ \partial_1^2-\partial_0\partial_2]$

Bad news: d>2 seems hard!

Ex. $D(C[x,y]^{Z/2Z}) =$
$C[x^2,xy,y^2,x\partial_x,x\partial_y,y\partial_x,\ y\partial_y,\ \partial_x^2,\ \partial_x\partial_y,\ \partial_y^2]$

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