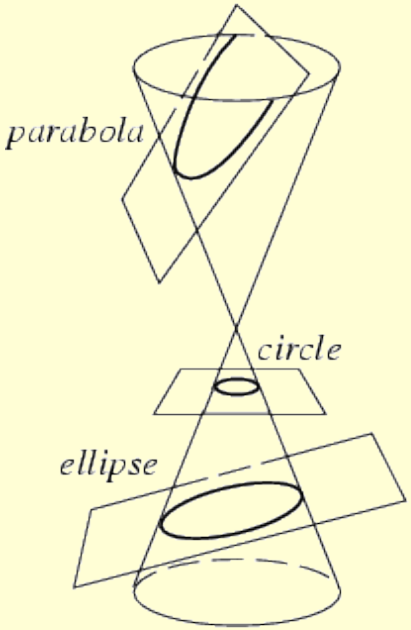
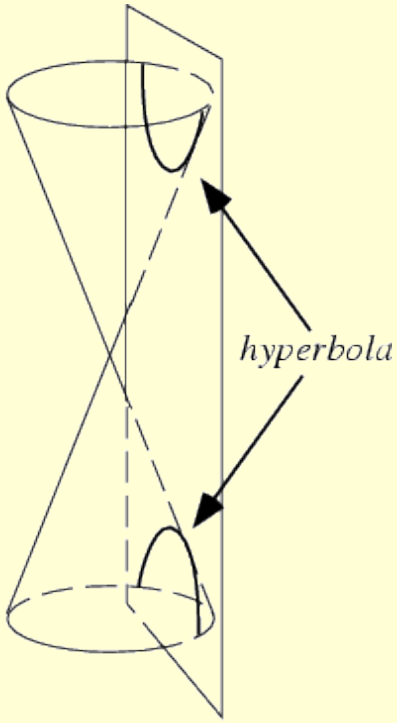
November 18, 2005
Howard University



Counting Conics

Will Traves
U.S. Naval Academy &
University of Maryland



USNA Trident Project

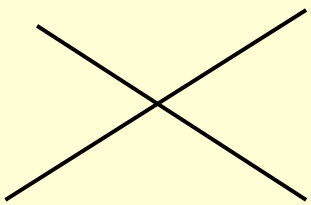
This talk reports on joint work with MIDN Andrew Bashelor and my colleague at USNA, Amy Ksir.

Our work grew out of Bashelor's Trident project, a full-year undergraduate research project focused on enumerative algebraic geometry.

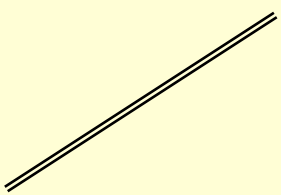
Plane Conics

Circles, **ellipses**, **parabolas** and **hyperbolas** are familiar examples of plane conics. For us, a plane conic will be the points satisfying a degree 2 polynomial relation:

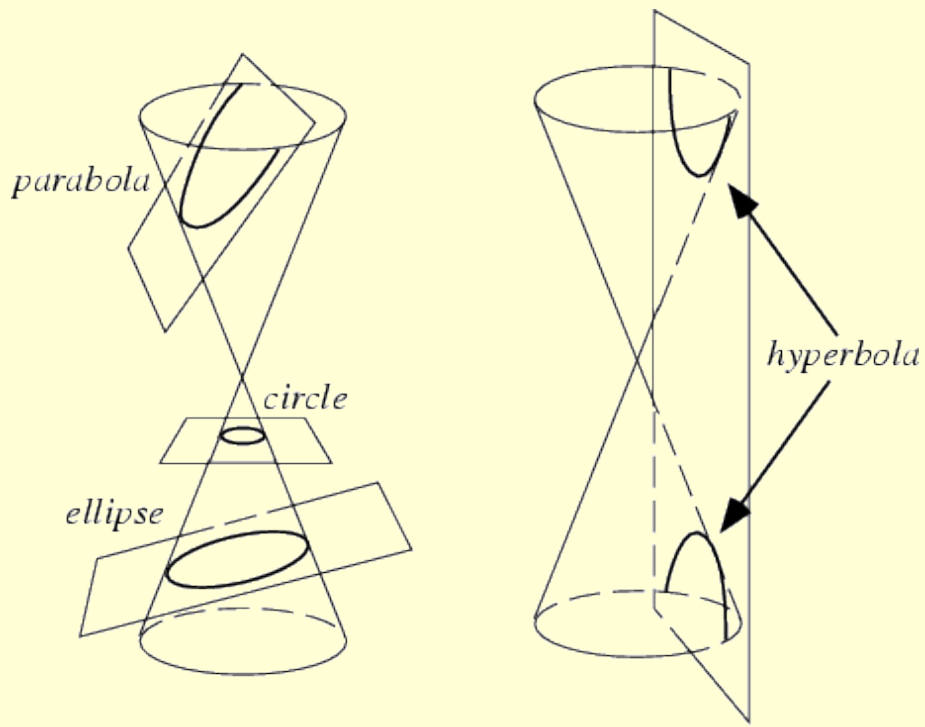
$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$



Degenerate conic
(2x+y-3)(3x-y+2)=0

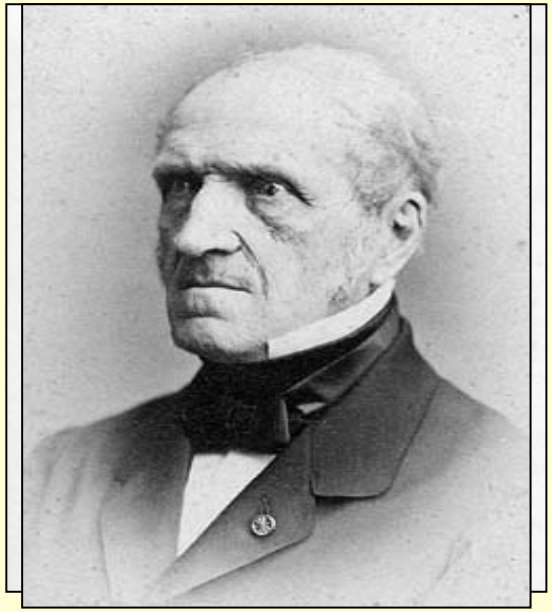


Double line
$(3x-y+2)^2 = 0$

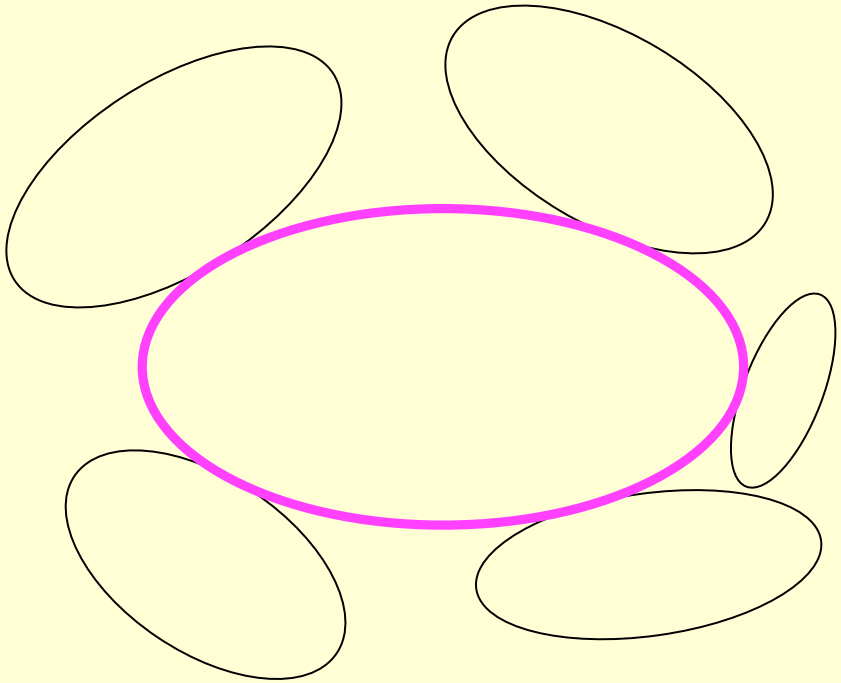


Steiner's Problem

How many conics are tangent to five given plane conics?

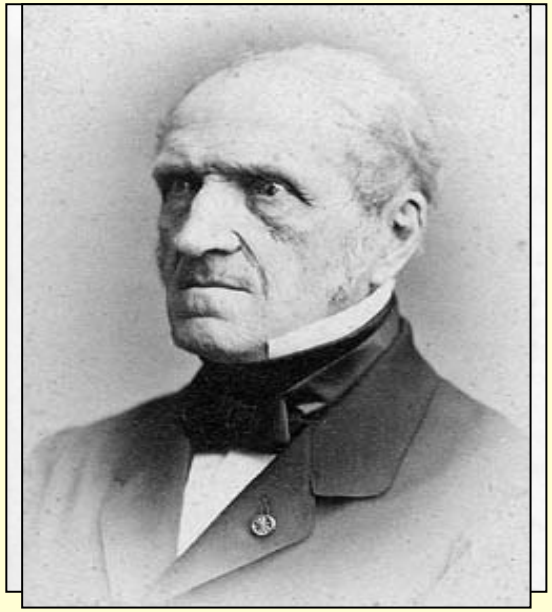


Michel Chasles
1793 – 1880



Steiner's Problem

How many conics are tangent to five given plane conics?

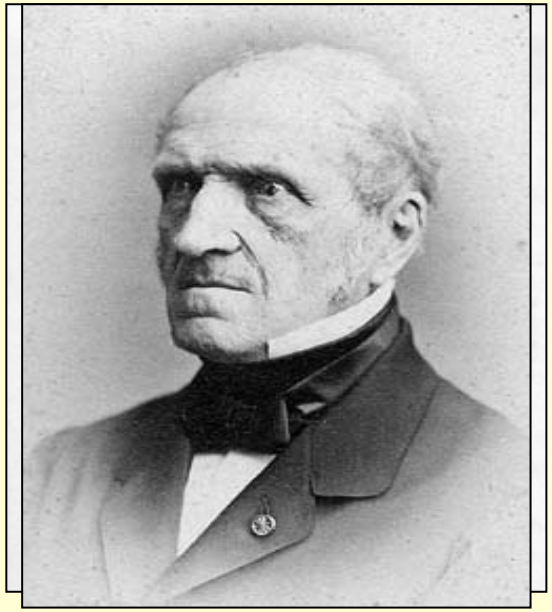


Michel Chasles
1793 – 1880

7776

Steiner's Problem

How many conics are tangent to five given plane conics?



Michel Chasles
1793 – 1880

A Parameter Space for Conics

Each conic is given by an equation,

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

but not uniquely so.

$$y - x^2 = 0 \quad \Leftrightarrow \quad 3y - 3x^2 = 0$$

If the point (a,b,c,d,e,f) represents the conic then so does (λa, λb, λc, λd, λe, λf). In fact, any point on the line spanned by the vector <a,b,c,d,e,f> represents the same conic.

Lines in R^6 form a **5-dimensional projective space**, $\mathbf{P}^5$, so the parameter space for conics is P^5.

We use the notation [a:b:c:d:e:f] to denote the conic. This reminds us that the values of the coordinates are less important than their ratios to one another.

Solving Enumerative Problems

Each condition in an enumerative problem imposes constraints on the conics that we need to count. These lead to subsets of our parameter space. We'll count the number of points in the intersection of these subsets.

This allows us to use both the geometry of conics in the plane and the geometry of P^5 to solve enumerative problems.

Example: H_p: set of conics passing through the point p.

For p(2,3), any conic passing though p must satisfy

$$a2^2 + b(2)(3) + c3^2 + d2 + e3 + f = 0$$

a linear condition. So the set of points H_p is a **hyperplane** in P^5.

Theorem: There is a unique conic passing through 5 points in general position (no four collinear).

Solving Enumerative Problems

Each condition in an enumerative problem imposes constraints on the conics that we need to count. These lead to subsets of our parameter space. We'll count the number of points in the intersection of these subsets.

This allows us to use both the geometry of conics in the plane and the geometry of P^5 to solve enumerative problems.

Example: H_p: set of conics passing through the point p.

For p(x,y), any conic passing though p must satisfy

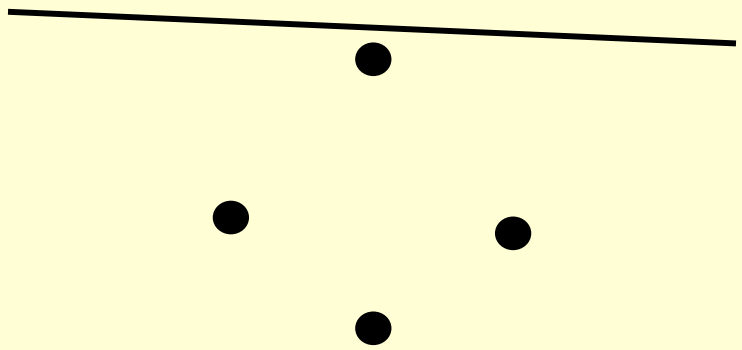
$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

a linear condition. So the set of points H_p is a **hyperplane** in P^5.

Theorem: There is a unique conic passing through 5 points in general position (no four collinear).

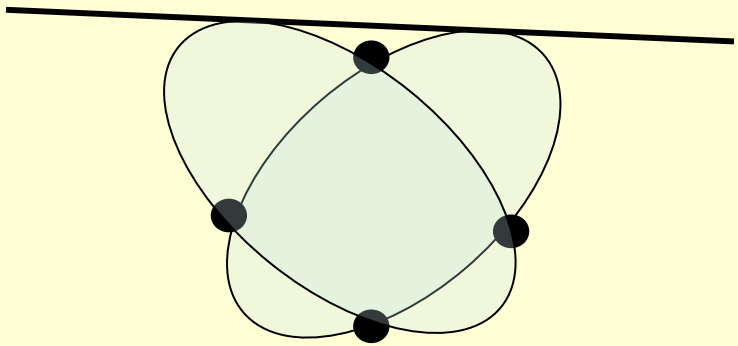
Conics Tangent to Lines

QUESTION: How many conics pass through 4 points and are tangent to a line in general position?



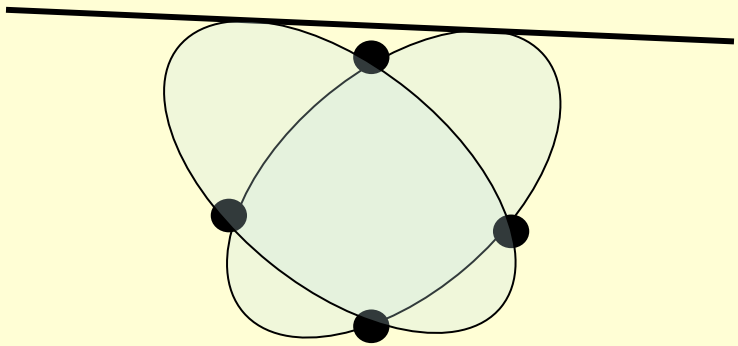
Conics Tangent to Lines

QUESTION: How many conics pass through 4 points and are tangent to a line in general position?



Conics Tangent to Lines

QUESTION: How many conics pass through 4 points and are tangent to a line in general position?

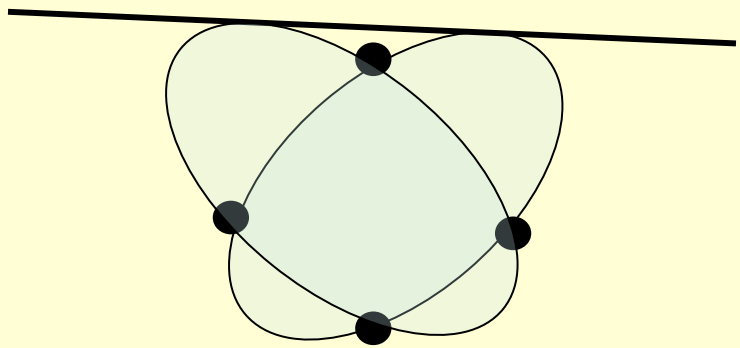


H_L: Conics tangent to line L given by **$y = Mx + B$**.

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

Conics Tangent to Lines

QUESTION: How many conics pass through 4 points and are tangent to a line in general position?



H_L: Conics tangent to line L given by **$y = Mx + B$**.

$$ax^2 + bx(Mx+B) + c(Mx+B)^2 + dx + e(Mx+B) + f = 0$$

Discriminant $b^2 - 4ac = 0$ $\Longleftrightarrow$ Equation has double root $\Longleftrightarrow$ Line is tangent to conic.

So H_L is a hypersurface in P^5 defined by a degree 2 equation.

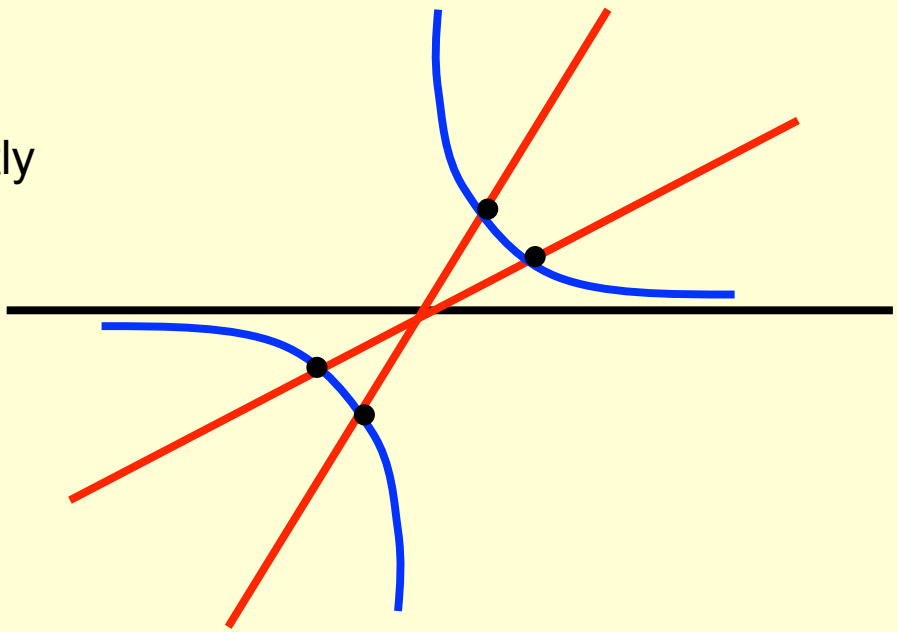
Passage to P^2

QUESTION: How many conics pass through 4 points and are tangent to a line in general position?

ANSWER: **Two.**

If we we work over the **complex numbers**, we always get the expected number of points in intersections (counted with multiplicity).

But we need to interpret tangency correctly and work in the **projective plane** P^2. This is like the Euclidean plane, but we attach a point at infinity for each direction.



Bézout's Theorem

Bézout's Theorem states that if we have **n hypersurfaces** (with no common component) of degrees $d_1, \ldots, d_n$ in P^n then their intersection consists of $\mathbf{d_1 d_2 \ldots d_n}$ points, counted with appropriate multiplicities.

This helps us answer questions like, "How many conics pass through 3 points and are tangent to 2 lines?"

ANSWER: Four.
The intersection of the corresponding hypersurfaces in P^5 consists of $(1)^3(2)^2 = 4$ points by Bézout's Theorem. These points correspond to the 4 conics.

How many conics pass through 2 points and are tangent to 3 lines? **8**
How many conics pass through 1 point and are tangent to 4 lines? **16**
How many conics are tangent to 5 lines? **32**

WARNING!!

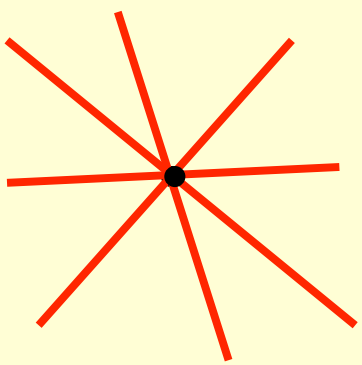
Duality

There is a duality between **points** and **lines** in P^2.

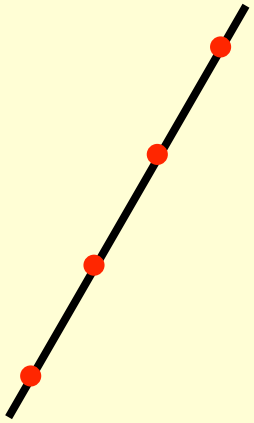
Point: $[a:b:c] \in P^2$ **Line** in P^2: $Ax+By+Cz=0$.

Lines are parameterized by tuples $[A:B:C]$ $\Longrightarrow$ $\{\text{Lines in } P^2\} \cong P^2$.

The set of lines passing through a point is a line in this dual P^2.
We have a duality map that exchanges points and lines in P^2 and the dual P^2.



Point [a:b:c]
Lines Ax+By+Cz=0



Line aX+bY+cZ=0
Points [A:B:C]

Duality for Conics

For each conic Q we have a dual conic, consisting of lines tangent to Q.

For instance, if Q is the conic $\mathbf{x^2 - yz = 0}$ then a line $\mathbf{Ax+By+Cz=0}$ with $A \neq 0$ meets Q where

$$\mathbf{A^2 x^2 - A^2 yz = 0} \text{ and } \mathbf{Ax = -(By+Cz)}$$

i.e. $\mathbf{(By+Cz)^2 - A^2 yz = 0}$

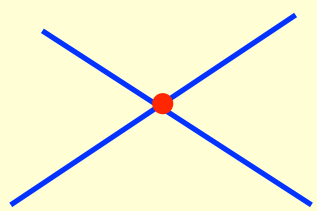
i.e. $\mathbf{B^2 y^2 + (2BC - A^2) yz + C^2 z^2 = 0.}$

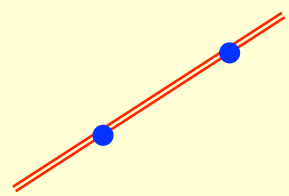
The line is tangent when the discriminant $A^2(A^2 - 4BC)$ vanishes, or when $\mathbf{A^2 - 4BC = 0}$.

The conic $ax^2 + by^2 + cxy + dxz + eyz + fz^2 = 0$ has dual conic given by

$$\mathbf{(e^2\text{-}4cf)A^2+(4bf\text{-}2de)AB+(d^2\text{-}4af)B^2+(4cd\text{-}2be)AC+(4ae\text{-}2bd)BC+(b^2\text{-}4ac)C^2 = 0.}$$

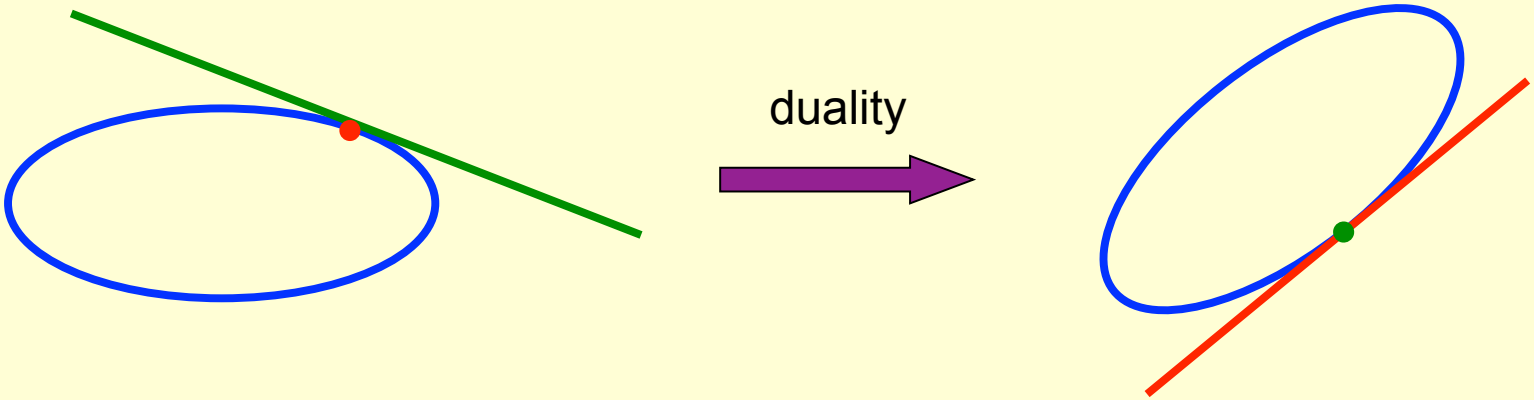
Degenerate conics dualize to double lines, but double lines do not have a well defined dual.





Duality and Tangency

Duality preserves incidence and tangency. If a line is tangent to a conic then after dualizing, the point lies on the dual conic. If a point lies on a conic then after dualizing, the corresponding line is tangent to the dual conic.



Thus **duality induces a symmetry** between our problems with points and lines. A conic passes through n points and is tangent to 5-n lines iff its dual passes Through the 5-n dual points and is tangent to the n dual lines.

So there are **4** conics through 2 points and tangent to 3 lines,
2 conics through 1 point and tangent to 4 lines,
and just **1** conic tangent to 5 lines.

Steiner's Problem

Question: How many conics are tangent to five general conics?

Steiner's answer involved the **hypersurface H_Q** of conics tangent to a fixed conic Q.

H_Q has degree 6:

- Conic **Q** can be parameterized by a **degree 2 map**

$$x^2 + y^2 = z^2$$
$$x = (s^2 - t^2)/2$$
$$y = st$$
$$z = (s^2 + t^2)/2.$$

- The 4 points of intersection of a general conic and **Q** have **(s,t)** coordinates given by a degree 4 polynomial.

- The conics are tangent when we have multiple roots.

- This happens when the discriminant of the polynomial is zero, a degree 6 condition.

Steiner's Problem

Question: How many conics are tangent to five general conics?

Steiner's answer involved the **hypersurface H_Q** of conics tangent to a fixed conic Q.

H_Q has degree 6.

Steiner reasoned that by Bézout's Theorem, there must be $\mathbf{6^5 = 7776}$ conics tangent to five general conics.

Unfortunately, every double line is tangent to every conic, so Bézout's Theorem does not apply (the H_Q's share a common component, the collection of double lines).

Blowing up the Veronese Surface

Duality saved us when we ran into excess intersection before, so it makes sense to study the duality map in more detail:

The duality map on conics is the map from P^5 to P^5 that sends the conic corresponding to [a:b:c:d:e:f] to the dual conic corresponding to
$[(e^2-4cf): (4bf-2de): (d^2-4af): (4cd-2be): (4ae-2bd): (b^2-4ac)]$.

This map is well-defined except for the **double line conics**, where all 6 components vanish. These cut out a surface, called the **Veronese surface**, V.

Algebraic geometers call such a partially-defined map a **morphism**. A morphism can be extended to an honest map by expanding the domain, using a technique called blowing up.

The **blow-up** of the Veronese is the closure of the graph of the duality map in $P^5 \times P^5$. The blow-up map π: $BL_V P^5 \rightarrow P^5$ is the projection onto the first P^5 factor.

The Exceptional Divisor

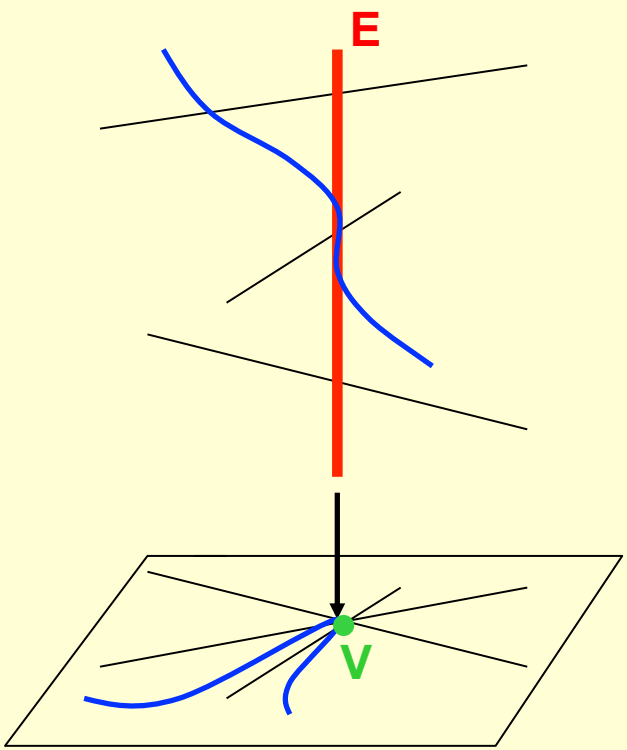
The **blow-up** of the Veronese is the closure of the graph of the duality map in $P^5 \times P^5$. The blow-up map π: $BL_V P^5 \rightarrow P^5$ is the projection onto the first P^5 factor.

Off the Veronese, $BL_V P^5$ and P^5 look the same (they are isomorphic), but $\pi^{-1}(V)$ is much bigger than V. In fact, $\pi^{-1}(V)$ is a hypersurface in $BL_V P^5$, called the **exceptional divisor** E.

Essentially, each point in V is replaced in $BL_V P^5$ by its **projective normal bundle**.

Blowing up causes hypersurfaces that intersected on the Veronese to become separated.

Blowing up also tends to smooth sharp corners, so it is often used to **desingularize** algebraic varieties.



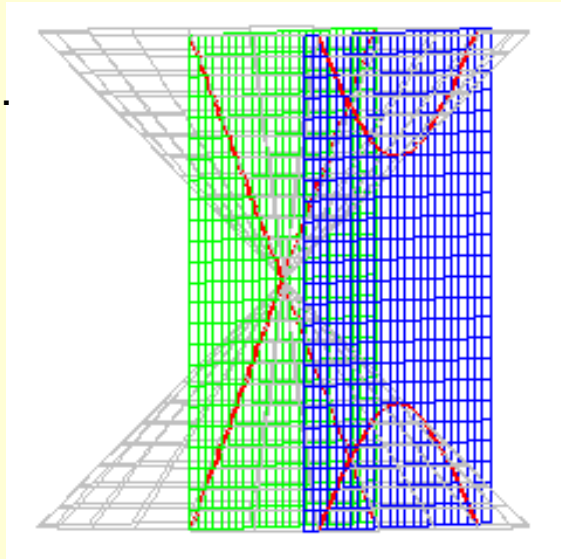
The Chow Ring

We'd like to solve our enumerative problem by intersecting varieties on the blow-up. To do this we want an analogue of Bézout's Theorem that holds on $BL_V P^5$. This is provided by the **Chow ring**, which is made up of rational equivalence classes of cycles.

A **k-cycle** on a variety X is just a finite formal sum of irreducible subvarieties of X with integer coefficients.

Both the zeros and poles of a rational function are cycles.

$$\frac{(X-Y)^2(Y-3Z)}{X^3+Y^3+Z^2}$$ has zeros on the cycle $2L_1 + L_2$.



We say that two k-cycles Y_1 and Y_2 are **rationally equivalent** if there is a k+1 cycle W containing both and a rational function f on W such that Y_1 = zeros(f) and Y_2 = poles(f).

We should think of rationally equivalent cycles as being **deformations** of one another.

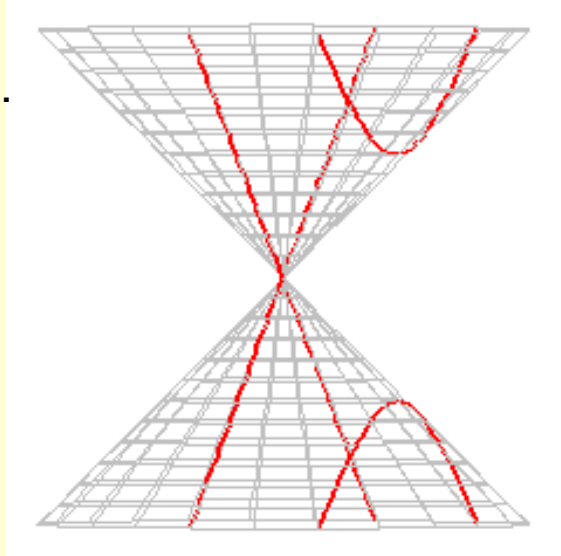
The Chow Ring

We'd like to solve our enumerative problem by intersecting varieties on the blow-up. To do this we want an analogue of Bézout's Theorem that holds on $BL_V P^5$. This is provided by the **Chow ring**, which is made up of rational equivalence classes of cycles.

A **k-cycle** on a variety X is just a finite formal sum of irreducible subvarieties of X with integer coefficients.

Both the zeros and poles of a rational function are cycles.

$$\frac{(X-Y)^2(Y-3Z)}{X^3+Y^3+Z^2}$$ has zeros on the cycle $2L_1 + L_2$.



We say that two k-cycles Y_1 and Y_2 are **rationally equivalent** if there is a k+1 cycle W containing both and a rational function f on W such that Y_1 = zeros(f) and Y_2 = poles(f).

We should think of rationally equivalent cycles as being **deformations** of one another.

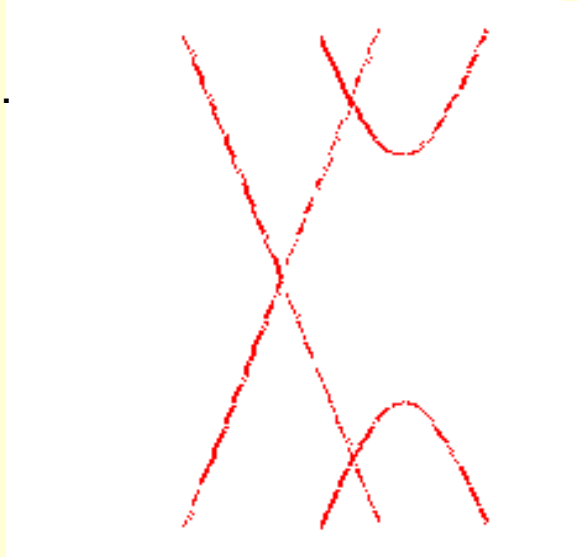
The Chow Ring

We'd like to solve our enumerative problem by intersecting varieties on the blow-up. To do this we want an analogue of Bézout's Theorem that holds on $BL_V P^5$. This is provided by the **Chow ring**, which is made up of rational equivalence classes of cycles.

A **k-cycle** on a variety X is just a finite formal sum of irreducible subvarieties of X with integer coefficients.

Both the zeros and poles of a rational function are cycles.

$$\frac{(X-Y)^2(Y-3Z)}{X^3+Y^3+Z^2}$$ has zeros on the cycle $2L_1 + L_2$.



We say that two k-cycles Y_1 and Y_2 are **rationally equivalent** if there is a k+1 cycle W containing both and a rational function f on W such that Y_1 = zeros(f) and Y_2 = poles(f).

We should think of rationally equivalent cycles as being **deformations** of one another.

Arithmetic in the Chow Ring

The **Chow ring** consists of rational equivalence classes of cycles in X.

We add classes formally, but **multiplication** corresponds to transverse **intersection**: $[Y_1] \bullet [Y_2] = [Y_1 \cap Y_2]$.

For $X = P^5$ the **Chow ring** consists of sums of [H] (a hyperplane class), $[H]^2$, $[H]^3$, $[H]^4$ (a line) and $[H]^5$ (a point).

Bézout's Theorem can be restated in terms of the Chow ring:
$[d_1H][d_2H]\ldots[d_5H] = d_1d_2 \ldots d_5 [H]^5 = d_1d_2\ldots d_5$ points.

Our strategy has been to use Bézout's Theorem to intersect hypersurfaces in P^5. But for Steiner's problem we need to work in BL_VP^5 in order to avoid **excess intersection** along the Veronese. We **blow up** to separate the H_Q's and then use the **Chow ring** to compute the number of points in their intersection.

Pullbacks

Most of the blow-up $X = BL_V P^5$ looks like P^5, the only new hypersurface class is the exceptional divisor **E** = $\pi^{-1}(V)$.

We can pull back cycles in P^5 to cycles in $BL_V P^5$. Each cycle Y pulls back to its strict transform plus some number of copies of E:

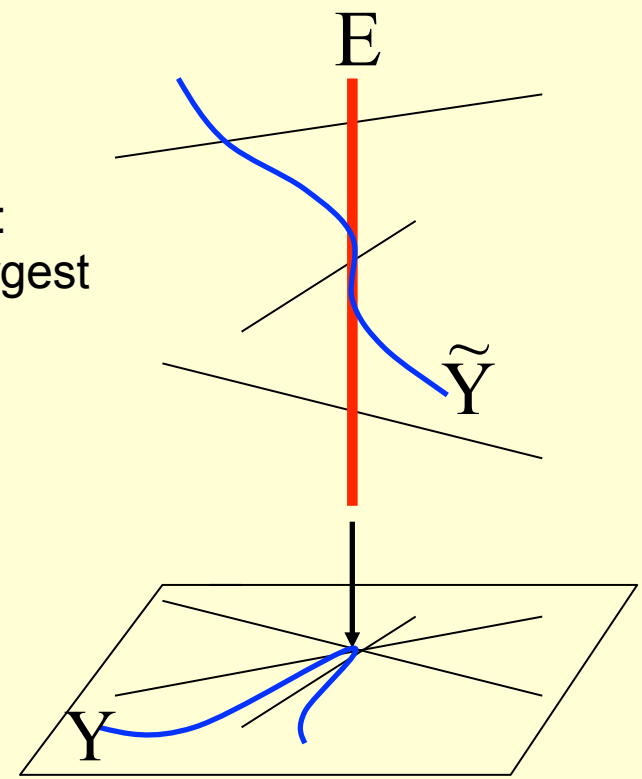
$$\pi^*(Y) = \tilde{Y} + nE$$

The number n measures the number of times Y contains $V = \pi(E)$. We can compute this using ideals: Y is defined by a single equation F=0 and n is the largest integer so that $F \in \mathbf{I}(V)^n$.

$$\tilde{H} = \boldsymbol{\pi}^*(\mathbf{H}_{\mathbb{P}}) = \tilde{\mathbf{H}}_{\mathbb{P}}$$

$$2\tilde{H} = \pi^*(H_L) = \tilde{H}_L + E$$

$$6\tilde{H} = \pi^*(H_Q) = \tilde{H}_Q + 2E$$



Computations on the Blow-up

After blowing up the total transform is $\pi^*(H_Q) = \tilde{H}_Q + 2E = 6\tilde{H}$

To answer Steiner's problem we must find $(\tilde{H}_Q)^5 = (6\tilde{H} - 2E)^5$

But this requires knowing the ring structure of the Chow ring for the blow-up.
Another way is to use the classes H_P and H_L that we defined earlier:

$$\pi^*(H_P) = \tilde{H}_P = \tilde{H}$$
$$\pi^*(H_L) = \tilde{H}_L + E = 2\tilde{H}$$

$$\Rightarrow$$

$$\tilde{H} = \tilde{H}_P$$
$$E = 2\tilde{H}_P - \tilde{H}_L$$
$$\tilde{H}_Q = 6\tilde{H} - 2E = 2\tilde{H}_P + 2\tilde{H}_L$$

It follows that

$$\begin{aligned}(\tilde{H}_Q)^5 &= (2\tilde{H}_P + 2\tilde{H}_L)^5 \\ &= 32(\tilde{H}_P^5 + 5\tilde{H}_P^4\tilde{H}_L^1 + 10\tilde{H}_P^3\tilde{H}_L^2 + 10\tilde{H}_P^2\tilde{H}_L^3 + 5\tilde{H}_P^1\tilde{H}_L^4 + \tilde{H}_L^5) \\ &= 32(1 + 5(2) + 10(4) + 10(4) + 5(2) + 1) \\ &= 3264.\end{aligned}$$

Summary & Conclusion

We showed how to answer enumerative questions about conics using techniques from algebraic geometry: **Bézout's Theorem, duality, blowing-up,** and the **Chow ring**.

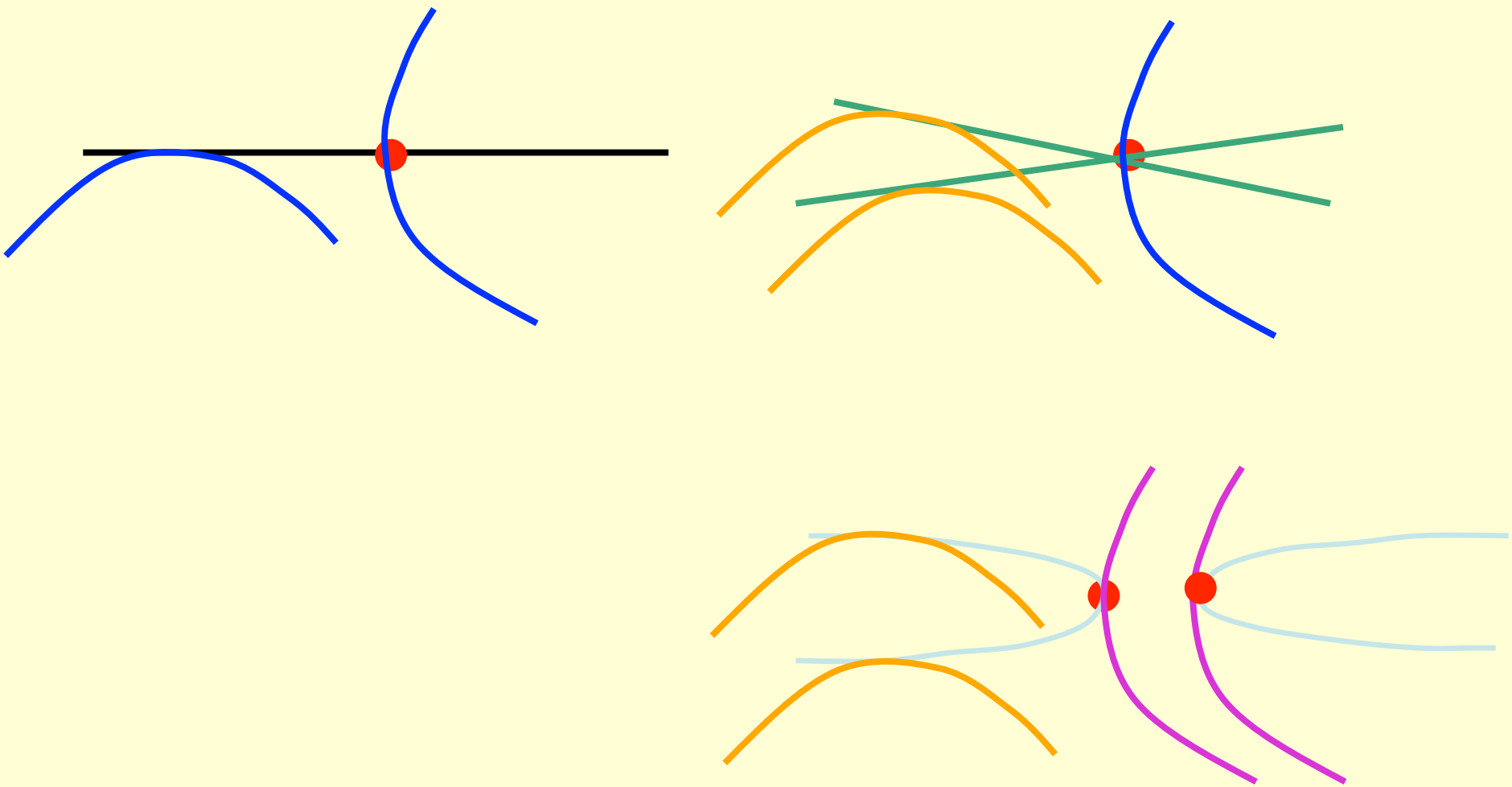
These techniques helped us avoid **excess intersection** and filter the **double line conics** from our answer.

There are many other interesting enumerative questions that also lead to nice mathematics. A problem in **Schubert Calculus** asks:

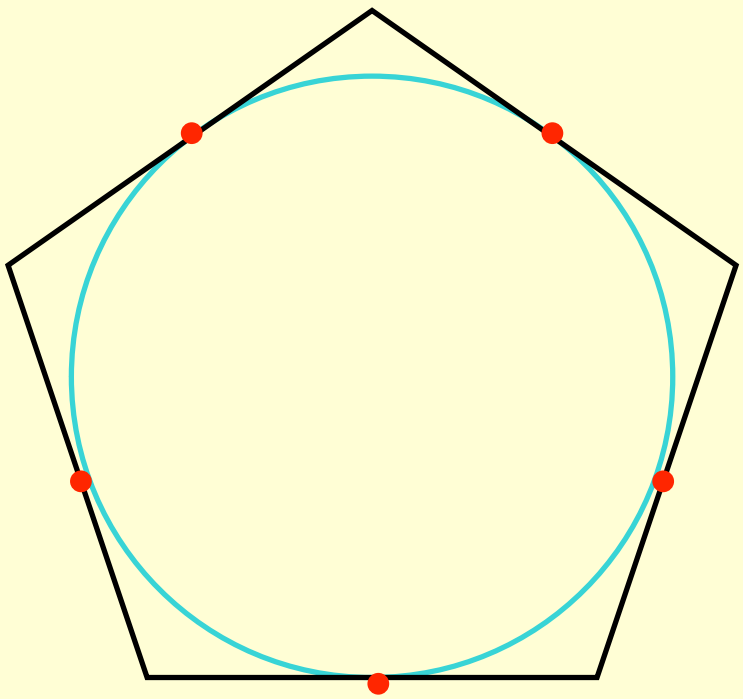
"How many lines in P^3 meet 4 general lines?"

The five-points theorem can be generalized by asking
"How many rational curves of degree d pass through 3d-1 general points?"
These are the **Gromov-Witten invariants** and play a role in **String theory**.
Kontsevich found a beautiful recurrence relation for these numbers.

$$\tilde{H}_Q = 2\tilde{H}_L + 2\tilde{H}_P$$



Explaining the Number 3264



There is only 1 conic tangent to any L of these lines and through P of these points (L+P = 5).

Nudging the points along the lines gives 102 conics.

Scissoring the double conics at the points and deforming to each to a hyperbola doubles the number of conics satisfying each condition, multiplying the number of conics by $2^5 = 32$.

$$(\widetilde{H}_Q)^5 = 32(\widetilde{H}_P^5 + 5\widetilde{H}_P^4\widetilde{H}_L^1 + 10\widetilde{H}_P^3\widetilde{H}_L^2 + 10\widetilde{H}_P^2\widetilde{H}_L^3 + 5\widetilde{H}_P^1\widetilde{H}_L^4 + \widetilde{H}_L^5) = 3264.$$