Higher Derivations in Algebraic Geometry

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Thank You.

Overview

- Invariant Theory of Graphs,
- Steenrod Algebra,
- Algebraic Geometry,
- Higher Derivations,
- Jet Spaces and Applications.

Question: Can we find invariants to distinguish graphs up to isomorphism?

Graphs(n): set of labeled graphs on n vertices.

Can we find a set of functions $f_1, \ldots, f_k$ that distinguish isomorphism classes of Graphs(n)? By this we mean:

- all functions f_i are constant on isomorphism classes

- if Γ_1 and Γ_2 are nonisomorphic graphs then there is a function f_i with $f_i(\Gamma_1) \neq f_i(\Gamma_2)$.

Graphs $\Gamma \in$ Graphs(n) are determined by their edges.

<u>Define</u>: edge functions e_{ij} : Graphs$(n) \to \mathbb{F}_2$ for $i < j$.

Polynomial ring: $R = \mathbb{F}_2[e_{12}, \ldots, e_{(n-1)n}]$

$G = S_n$ acts on Graphs(n) by permuting the vertex labels.

Induced action on the edge functions: $\sigma(e_{ij}) = e_{\sigma(i)\sigma(j)}$.

Invariant functions, $R^G \subseteq R$, are finitely generated (Hilbert's Basis Theorem) and separate isomorphism classes.

Example 1 *There are 8 labeled graphs on 3 vertices. However, there are only four isomorphism types of such graphs (in this simple case, they are determined by the number of edges in the graph). The invariant functions*

$$R^G = \mathbb{F}_2[e_{12} + e_{13} + e_{23},\ e_{12}e_{13} + e_{12}e_{23} + e_{13}e_{23},\ e_{12}e_{13}e_{23}]$$

separate Graphs(3).

Here our polynomial ring was defined over a finite field $\mathbb{F}_2$.

19th century: Invariant Theory developed for fields of characteristic zero.

In prime characteristic we have the Frobenius map $r \to r^p$.

Frobenius map is closely linked to the Steenrod Algebra.

Algebraic Approach to Steenrod Algebra: L. Smith and R. Wood.

The Steenrod Algebra

Notation: $R = k[x_1, \ldots, x_n]$, $k = \mathbb{F}_q$, $q = p^e > 0$.

Define: $\phi : R \to R[[t]]$ on each x_i via

$$x_i \mapsto x_i + x_i^q t$$

and extend ϕ to a ring homomorphism. For example,

$$x_i x_j \mapsto (x_i x_j) + (x_i^q x_j + x_i x_j^q)t + (x_i^q x_j^q)t^2.$$

Define operators $Q_i : R \to R$ to be the k-linear maps defined by applying ϕ and then extracting the coefficient of t_i from the resulting polynomial. For instance,

$$Q_1(x_i x_j) = x_i^q x_j + x_i x_j^q.$$

The operators Q_i satisfy a product rule that is an extension of the rule we teach in Calculus:

Theorem 2 (Cartan's Formula) $Q_k(fg) = \sum_{i+j=k} Q_i(f)Q_j(g)$.

Ex. $Q_1(x_i x_j) = Q_1(x_i)Q_0(x_j) + Q_0(x_i)Q_1(x_j) = x_i^q x_j + x_i x_j^q$.

$Gl(n, \mathbb{F}_q)$ acts on R by linear change of variables. This action commutes with the Q_i.

[Reason: action of g on $\{x_1, \ldots, x_n\}$ represented by same matrix as action on $\{x_1^q, \ldots, x_n^q\}$.]

Theorem 3 *The operators Q_i commute with $Gl(n, \mathbb{F}_q)$ action.*

If G is any group that acts linearly on R, then the Steenrod operations Q_i induce maps $R^G \to R^G$: if $r \in R^G$ and $g \in G$, then $g \cdot Q_i(r) = Q_i(g \cdot r) = Q_i(r)$.

The Q_i raise degree and preserve invariants. So they can be used to create new (higher degree) invariants from known invariants.

Research on the Steenrod Algebra

- Generators and Relations (the Adem relations).

- Structure of R^G as a module over the Steenrod Algebra.

- Interpretation of the Steenrod Algebra in terms of differential operators.

Algebraic Geometry

Algebraic geometers think of rings in a geometric way (and sometimes use algebra to formalize notions in geometry).

To explain this, consider polynomial functions on $\mathbb{C}^n$.

These can be added and multiplied, so they form a ring, $R = \mathbb{C}[x_1, \ldots, x_n]$.

<u>Def</u>: A variety is the common zero set of a collection of polynomial equations, $f_1 = 0, \ldots, f_k = 0$.

• The union of the two axes in $\mathbb{C}^2$ is just the zero set of $x_1 x_2$.

• The variety $X = \mathbb{V}(x_2 - x_1^2, x_2 - 2)$ is just the intersection of a parabola with a line – two points.

What are the functions on a variety V?

Polynomials in $(f_1, \ldots, f_k)$ restrict to zero on V so the ring of functions on V is

$$\mathbb{C}[V] = \frac{\mathbb{C}[x_1, \ldots, x_n]}{(f_1, \ldots, f_k)}.$$

This association is really an antiequivalence of categories: polynomial maps of varieties

$$\pi : X \to Y$$

correspond to ring homomorphisms

$$\mathbb{C}[Y] \to \mathbb{C}[X]$$

where $f \in \mathbb{C}[Y]$ is sent to $f \circ \pi$.

Rings also give rise to geometric objects: $R \mapsto Spec(R)$.

Maximal ideals in $\mathbb{C}[V]$ correspond to points on V. [But $Spec(R)$ contains more points too!]

The Ring $\mathbb{C}[[t]]$

Question: What is the geometric interpretation of the ring $\mathbb{C}[[t]]$?

There is only one maximal ideal (t) in this ring:
$Spec(\mathbb{C}[[t]]) = pt$????

The ring consists of formal power series centered at zero. Convergence in a microlocal neighbourhood of zero.

We think of $Spec(\mathbb{C}[[t]])$ as a point, together with a infintesimal tangent vector, an infintesimal second-order tangent vector, etc.

The image of a map $Spec(\mathbb{C}[[t]]) \to X$ is then just a formal arc in X: a point in X plus a microlocal curve through the point.

Higher Derivations

Let $R = k[x_1, \ldots, x_n]/I$ be a finitely generated ring.

Definition: A higher derivation from R to itself is an infinite collection of maps of k-algebras $\{D_0 = id_R, D_1, D_2, \ldots\}$ from R to R that patch together using the product rule:

$$D_k(fg) = \sum_{i+j=k} D_i(f)D_j(g).$$

Examples

Example 4 *The Steenrod operators* $\{Q_0, Q_1, \dots\}$ *determine a higher derivation from* $R = k[x_1, \dots, x_n]$ *to itself.*

Example 5 *In characteristic zero, any derivation* d *on* R *determines a higher derivation*

$$D_k = \frac{1}{k!} d^k.$$

For instance, the derivation $\frac{d}{dx}$ *on* $k[x]$ *induces a higher derivation on the polynomial ring.*

Higher derivations are supposed to extend our notion of derivations (maps satisfying the usual product rule $D(fg) = fD(g) + D(f)g$).

The derivations are represented by the module $\Omega_{R/k}$ of k-differentials on R.

Each derivation $D : R \to R$ is determined by $\phi : \Omega_{R/k} \to R$ such that $\phi \circ d = D$.

There is a similar construction that produces a k-algebra $HS_{R/k}$ representing the higher derivations.

There is a sequence of maps $(d_0, d_1, \dots)$ from R to $HS_{R/k}$ such that each higher derivation $\{D_0, D_1, \dots\}$ from R to R is determined by a unique map $\phi : HS_{R/k} \to R$ of k-algebras via $\phi \circ d_i = D_i$.

Higher Derivations to k

We can also consider higher derivations from R to k.

The whole theory goes through as before: higher derivations are collections of k-algebra maps

$$D_i : R \to k$$

with

$$D_k(fg) = \sum_{i+j=k} D_i(f)D_j(g).$$

These are once again determined uniquely by a k-algebra map $\phi : HS_{R/k} \to k$.

So we see that the collection of higher derivations $Der_k(R, k)$ is isomorphic to $Hom_{k-alg}(HS_{R/k}, k)$.

On the other hand, each higher derivation $\{D_0, D_1, \dots\}$ from R to k determines a map of k-algebras $\phi : R \to k[[t]]$ given by:

$$\phi(r) = D_0(r) + D_1(r)t + D_2(r)t^2 + \cdots$$

This ring homomorphism is determined by the images $\phi(x_i)$ of the coordinate functions x_i.

We require that $f(\phi(x_1), \dots, \phi(x_n)) = 0$ for all polynomials f in the defining ideal I.

So $Der_k(R, k)$ is isomorphic to $Hom_{k-alg}(R, k[[t]])$.

$Der_k(R,k)$ is isomorphic to $Hom_{k-alg}(HS_{R/k},k)$.

$Der_k(R,k)$ is isomorphic to $Hom_{k-alg}(R,k[[t]])$.

Thus: $Hom_{k-alg}(R,k[[t]]) \cong Hom_{k-alg}(HS_{R/k},k)$.

Taking Spec's:

$$[Spec(k[[t]]) \to Spec(R)] \cong [Spec(k) \to SpecHS_{R/k}] \cong SpecHS_{R/k}.$$

Jet space $J(Spec(R)) := Spec(HS_{R/k})$ parameterizes arcs on $Spec(R)$.

The Jet Space over X

Natural map $\pi : J(Spec(R)) \to Spec(R)$ sends each arc to the closed point it passes through.

Each arc γ gives rise to a map $R \to k[[t]]$.

This assigns a power series $a_i + HOT(t)$ to each coordinate function x_i.

Just looking at the constant terms of these power series gives a point $\pi(\gamma) = (a_1, \ldots, a_n)$ in k^n that lies on $X = Spec(R)$.

$J(X)_O :=$ the set of arcs through $O = \pi^{-1}(O)$.

Applications of Jet Spaces

- Nash's Conjecture
- Motivic Integration

Nash's Conjecture

Jet Space of $X \leftrightarrow$ resolution of singularities of X.

X: surface with an isolated singular point O.

$\bar{X}$: minimal resolution of singularities of X.

$\gamma : \bar{X} \to X$ gives rise to essential exceptional divisors

$$\pi^{-1}(O) = \cup E_i.$$

Nash: components of $J(X)_O \hookrightarrow \{E_i\}$.

Nash conjectured that this is a bijection.

FALSE: Kollar and Ishii.

Motivic Integration

Batyrev's Conjecture: two birationally equivalent Calabi-Yau manifolds have the same Hodge numbers.

Hodge numbers: $(h_{p,q} = dim H^p(\Omega^q, X))$ are numerical invariants.

Map: $X \mapsto \sum_{i,j} h_{p,q} u^p v^q \in \mathbb{Z}[u, v]$

Kontsevich: theory of motivic integration.

Map above factors through another map (motivic integration): $\{varieties\} \to \mathcal{M}$.

Plan: Show birational C-Y map to same elt of $\mathcal{M}$.

Theory of Integration

- Space on which to integrate ($J(X)$).
- Integrable sets (cylinder sets).
- Value set for integration: motivic ring $\mathcal{M}$.
- Change of variables formula: key to Batyrev's conjecture.

Other Applications of Motivic Integration

- zeta functions
- p-adic integration
- string theory
- mirror symmetry
- characterizations of singularities (Berkeley seminar)