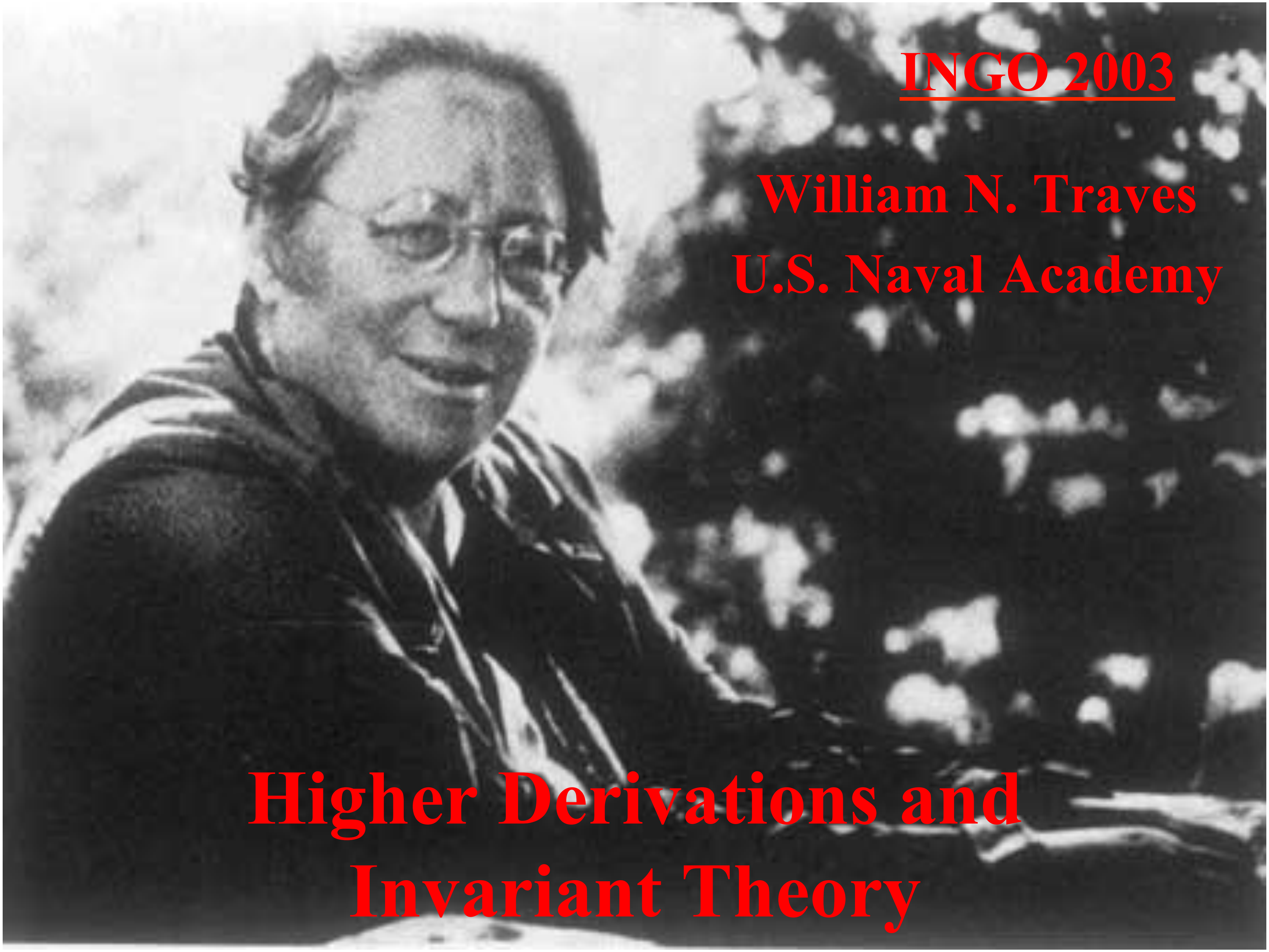

INGO 2003

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# Higher Derivations and Invariant Theory

Thank You.

# Outline

1) Invariant Theory and the Steenrod Algebra
2) Rings of Differential Operators
3) Higher Derivations
4) Jet Spaces and Applications

# Invariant Theory

- Invariant theory of 19th and 20th centuries focused on characteristic zero and nonmodular cases.

- Characteristic $p>0$ largely an afterthought – as in commutative algebra more generally.

- But prime characteristic methods are increasingly important.
    - Applications of commutative algebra to combinatorics.
    - New tool: Frobenius map.
        - Algebraic theory of tight closure
            - mimics and extends results from analysis
        - Invariant theory: Steenrod Algebra.

# The Steenrod Algebra

My thanks to Reg Wood for several nice lectures on the Steenrod algebra.

Like Reg, I will consider the Steenrod algebra from an algebraic point of view (as in Larry Smith's book).

Fix some notation:

$k = GF(q) = F_q$ $\quad q = p^s$ $\quad R=k[x_1, \ldots, x_n]$

G: subgroup of GL(n,k) acting linearly on R.

# Steenrod Algebra from $\varphi$

$R = k[x_1, \ldots, x_n]$ (= $k[x,y]$ or $k[x,y,z]$)

$\varphi: R \to R[[t]]$
$x_i \mapsto x_i + x_i^q t$

$\varphi(xy) = (x+x^q t)(y + y^q t) = xy + (xy^q + x^q y)\, t + x^q y^q\, t^2$

$Q_i : R \to R$ are the $i^{th}$ Steenrod operators obtained by applying $\varphi$ and extracting the coefficient of $t^i$

$Q_1(xy) = xy^q + x^q y$

A := Steenrod Algebra – the k-algebra generated by the $Q_i$.

# Properties of the Operators

**Cartan Formula:** $Q_k(fg) = \sum_{i+j=k} Q_i(f)Q_j(g)$

**Instability:** $Q_k(f_d) = \begin{cases} f^q & \text{if } k = d \\ 0 & \text{if } k > d \end{cases}$

Example: $Q_1(xy) = Q_0(x)Q_1(y) + Q_1(x)Q_0(y)$
$= xy^q + x^q y$

None of the operators are zero, though they are all nilpotent.

# The $Q_i$ Commute with G

The Steenrod operators $Q_i$ commute with the group action (linear change of variables).

So the Steenrod operators induce maps $Q_i$: $R^G \rightarrow R^G = k[A^n/G]$
$g \,.\, Q_i(r) = Q_i(g \,.\, r) = Q_i(r)$

The $Q_i$ raise degree and preserve invariants so they create new (higher degree) invariants from old.

# $Q_i$ and Frobenius

The $Q_i$ also satisfy an interesting relation with regard to the Frobenius map:

$$Q_i(r^{p^e}) = \begin{cases} (Q_{i/p^e}(r))^{p^e} & p^e \mid i, \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} r^{p^e} + Q_1(r^{p^e})t + Q_2(r^{p^e})t^2 + \cdots &= \varphi(r^{p^e}) \\ &= (\varphi(r))^{p^e} \\ &= (r + Q_1(r)t + Q_2(r)t^2 + \cdots)^{p^e} \\ &= r^{p^e} + Q_1(r)^{p^e} t^{p^e} + Q_2(r)^{p^e} t^{2p^e} + \cdots \end{aligned}$$

Equating coefficients of *t* gives the result.

# $R^{p^e}$-linearity of the $Q_i$

The $Q_i$ also satisfy an interesting relation with regard to the Frobenius map:

$$Q_i(r^{p^e}) = \begin{cases} (Q_{i/p^e}(r))^{p^e} & p^e \mid i, \\ 0 & \text{otherwise.} \end{cases}$$

Now for $p^e > i$ we have:

$$\begin{aligned} Q_i(r^{p^e} f) &= \sum_{m+n=i} Q_m(r^{p^e}) Q_n(f) \\ &= Q_0(r^{p^e}) Q_i(f) \\ &= r^{p^e} Q_i(f) \end{aligned}$$

# Steenrod Algebra: Structure

- Complete set of invariants – the Adem relations – is known.

- These are encoded by the Bullett-Macdonald identity.

- Much is also known about the structure of $R^G$ as a module over the Steenrod Algebra
    - Invariant ideals in $R^G$ – e.g. radical of a stable ideal is stable
    - For example, when $G = GL(n,k)$, there are only finitely many stable prime ideals in $R^G$ and these are generated by intervals in the Dixon invariants

- The Steenrod algebra can also be interpreted as a subring of the ring of differential operators on $R^G$.

# Rings of Differential Operators

Grothendieck: defined differential operators in an abstract way – subring of End(R) satisfying certain iterated commutator relations.

Case $R = \mathbf{C}[x_1,\ldots,x_n]$:

$$D(R) = \mathbf{C}[x_1, \ldots, x_n, d_1, \ldots, d_n]$$

generators satisfy the product rule:

$$[d_i, x_j] = d_i\, x_j - x_j\, d_i = \delta_{ij}$$

In this case, D(R) is the Weyl Algebra (see Coutinho's nice book).

# Problems in Characteristic p>0

Julia Hartmann mentioned some of the problems with differential operators in prime characteristic:

$d_1(x^p) = p\, x^{p-1} = 0$

Even worse: $d_1^p = 0$

Introduce the divided powers operators: $\quad d_i^{\,k} = \dfrac{1}{k!}\dfrac{\partial^k}{\partial x_i^k}$

Then set $D(k[x_1, ..., x_n]) = k[x_i, d_i^m]_{m>0}$

# Differential Operators on $R^G$

We are in the case where G is reductive so think of $R^G$ as S/I.

$$D(R^G) = D(S/I) = \frac{\{\theta \in D(S) : \theta(I) \subseteq I\}}{I\,D(S)}.$$

Remark 1: With this definition, it is not clear that $D(R^G)$ enjoys any nice properties.

Remark 2: We could also define $D(R_G)$ in this way.

# $R^{p^e}$-linearity of D(R)

Theorem (K.E. Smith): In characteristic p > 0, the ring of differential operators on a ring R is just the algebra of maps R → R that are $R^{p^e}$-linear for some power $p^e$.

Cor: The Steenrod algebra is a subalgebra of $D(R^G)$.

In fact, we can write the Steenrod operators as

$$Q_i = \sum_{|a|=i} x^{qa} d^a$$

$$= \sum_{a_1+\cdots+a_n=i} x_1^{qa_1} \cdots x_n^{qa_n} \frac{1}{a_1!\cdots a_n!} \frac{\partial^i}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}}$$

# Applications

Rings of differential operators find application in a wide variety of mathematical fields:

- Model quantum mechanics
- Used to study symplectic manifolds
- Local cohomology modules are finite over D(R)
- Close – but mysterious – connections to tight closure

# $D(R^G)$ and $D(R)^G$

The group G acts on R and this action extends to operators:

$$g \in G, \mathrm{d} \in \mathrm{D(R)} \Rightarrow (\mathrm{gd})(\mathrm{r}) = \mathrm{gd}(\mathrm{g}^{-1}\mathrm{r})$$

Note that if gd = d then d defines an operator on $R^G$:

$$g \in G, \mathrm{r} \in \mathrm{R}^G \Rightarrow \mathrm{gd}(\mathrm{r}) = \mathrm{gd}(\mathrm{g}^{-1}\mathrm{r}) = (\mathrm{gd})(\mathrm{r}) = \mathrm{d}(\mathrm{r})$$

Natural map: $D(R)^G \rightarrow D(R^G)$

# Questions about $D(R^G)$

- When is $D(R^G)$ finitely generated, or left or right Noetherian?
- When is $D(R^G)$ a simple ring?
- When is $R^G$ a simple module over $D(R^G)$?
- What about the same questions for $GrD(R^G)$?
- When is the map $D(R)^G \to D(R^G)$ surjective?

# Answers: G finite

Characteristic zero.

Kantor and Levasseur: $D(R^G)$ is

- finitely generated,

- left and right Noetherian.

- moreover: $D(R)^G \to D(R^G)$ is a surjection whenever G contains no pseudoreflections.

# Answers: Classical Groups

Schwarz, Levasseur, Van den Bergh, Musson, Stafford and many others have studied the classical groups acting on a polynomial ring in characteristic zero.

In most cases $D(R^G)$ – and even $GrD(R^G)$ – are finitely generated and left and right Noetherian. In all these cases, the map $D(R)^G \rightarrow D(R^G)$ is surjective. For example, this holds for tori, O(n), etc.

However, Schwarz has shown that there are representationsof $Sl_2(C)$ for which the map $D(R)^G \rightarrow D(R^G)$ is not surjective.

# Simplicity

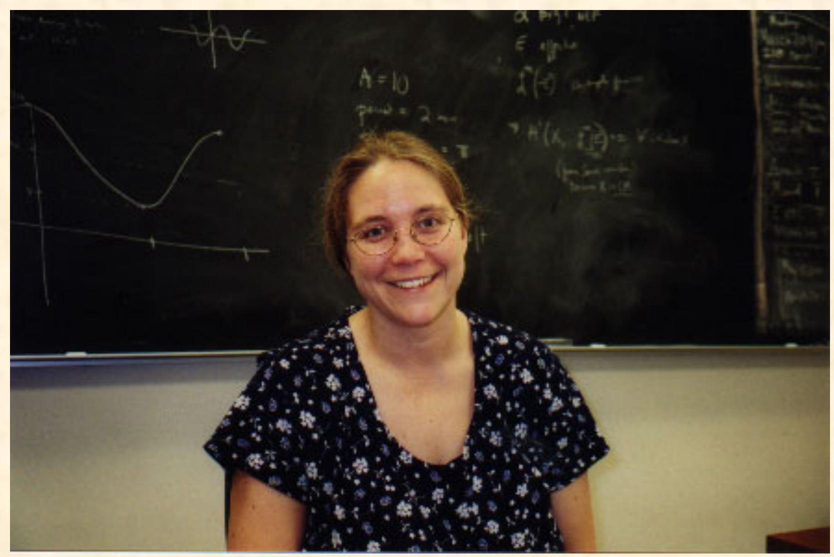
If D(S) is simple then the ring S is simple as a D(S)-module.

Proof:

- If I is a nonzero stable ideal in S then S/I is a D(S)-module.
- $\mathrm{Ann}_{D(S)}(S/I)$ is a two-sided ideal in D(S) that contains I.
- So $\mathrm{Ann}_{D(S)}(S/I) = D(S)$.
- Thus $S/I = 1(S/I) = 0$ and $I = S$.
- So S contains no proper D(S)-modules.

General feeling: characteristic p is harder than char. 0

<u>Theorem</u> (K.E. Smith and Van den Bergh): In prime characteristic $D(R^G)$ is always a simple ring.



# Higher Derivations

The Steenrod operations are a example of a higher derivation, collections of operators that generalize the behavior of derivations on commutative rings.

Let $R = k[x_1, ..., x_n]/I$

Definition: A higher derivation from R to R is an infinite collection of k-algebra maps $\{D_0 = id_R, D_1, D_2, ...\}$ from R to R that patch together using the product rule

$$D_k(fg) = \sum_{i+j=k} D_i(f)D_j(g)$$

# Examples of Higher Derivations

(1) The Steenrod operators $\{Q_0, Q_1, \ldots\}$ determine a higher derivation from $R=k[x_1, \ldots, x_n]$ to itself.

(2) In characteristic zero, any derivation d on R determines a higher derivation

$$D_k = \frac{1}{k!} d^k$$

For instance, the derivation d/dx on k[x] induces a higher derivation on the polynomial ring.

# Exponential Maps

Each higher derivation $\{D_0, D_1, D_2, \ldots\}$ from R to R gives rise to a map of k-algebras

$$\begin{aligned} \varphi: R &\to R[[t]] \\ r &\mapsto D_0(r) + D_1(r)\,t + D_2(r)\,t^2 + \ldots. \end{aligned}$$

The product rule guarantees that this map is a ring map.

There is no instability result for higher derivations, but each $D_i$ is $R^{p^e}$-linear for some power $p^e$. So each higher derivation is a differential operator.

# The Higher Derivation Algebra

The higher derivation algebra *HDer*(R) on a ring R is just the R-algebra generated by the components of all higher derivations on R.

Larry Smith asked whether A = HDer.

$$R^G \otimes A \subseteq HDer(R^G) \subseteq D(R^G)$$

# Case: $R^G$ a Polynomial Algebra

Here $\mathrm{HDer}(R^G) = D(R^G)$ but $R^G A \neq \mathrm{Hder}(R^G)$.

Equality follows from direct calculation. In fact $\mathrm{HDer}(S) = D(S)$ whenever S is smooth over k.

Inequality now follows because $R^G$ has A-stable ideals (for example, the augmentation ideal), but is $D(R^G)$-simple.

# Nakai's Conjecture

Conjecture: S is smooth over k if and only if
HDer(S) = D(S).

Ishibashi proved Nakai's conjecture for $R^G$, G a finite group.

When $R^G$ is singular, there is a nice theory of $HDer(R^G)$-stable ideals, similar to that developed by Smith for A-stable ideals. But it remains open whether $HDer(R^G) = R^G A$ in the singular case.

# Derivations are Representable

$$
\begin{tikzcd}
 & \Omega_{S/k} \arrow[d, dashed, "\varphi\text{: S-module map}"] \\
S \arrow[ur, "d"] \arrow[r, "D"'] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
 & HS_{S/k} \arrow[d, dashed, "\varphi\text{: k-algebra map}"] \\
S \arrow[ur, "\underline{d}"] \arrow[r, "\underline{D}=\{D_0,D_1,\ldots\}"'] & k
\end{tikzcd}
$$

Higher derivations $(S \Rightarrow S) \cong \mathrm{Hom}_{k\text{-alg}}(HS_{S/k}, S)$

$$\{D_0, D_1, \ldots\} \mapsto \varphi : \forall i \;\; D_i = \varphi \circ d_i$$

# Arcs

Suppose $S=k[x_1,\ldots,x_n]/I$ and $\underline{D}=\{D_0, D_1, \ldots\}$ is a higher derivation from S to k.

Then we get a ring map $\varphi$: $S \rightarrow k[[t]]$ given by
$\varphi(s) = D_0(s) + D_1(s)t + D_2(s)t^2 + \ldots.$

This map is determined by the images $\varphi(x_i)$. These need to satisfy $f(\varphi(x_1),\ldots, \varphi(x_n))=0$ for each f in the defining ideal I.

$$\text{Higher derivations}\,(S \to k) \cong \ \mathrm{Hom}_{\text{k-alg}}(S, k[[t]])$$

# An Adjointness Result

$$\text{Theorem} : \mathrm{Hom}_{\text{k-alg}}(HS_{S/k}, k) \cong \mathrm{Hom}_{\text{k-alg}}(\mathrm{S}, k[[t]])$$

Taking Spec's:

$$\begin{aligned}[\text{maps } \mathrm{Spec}(\mathrm{k}[[\mathrm{t}]]) \to \mathrm{Spec}(\mathrm{S})] &\cong [\text{maps } \mathrm{Spec}(\mathrm{k}) \to \mathrm{Spec}(\mathrm{HS}_{\mathrm{S/k}})] \\ &\cong \mathrm{Spec}(\mathrm{HS}_{\mathrm{S/k}})\end{aligned}$$

So $\mathrm{Spec}(\mathrm{HS}_{\mathrm{S/k}})$ parameterizes arcs on $\mathrm{Spec}(\mathrm{S})$.

# The Jet Space

The Jet Space $J(S) = \mathrm{Spec}(HS_{S/k})$ parameterizes arcs on Spec(S).

We have a map *pr* : $J(S) \rightarrow \mathrm{Spec}(S)$ that sends each arc to the point it passes through.

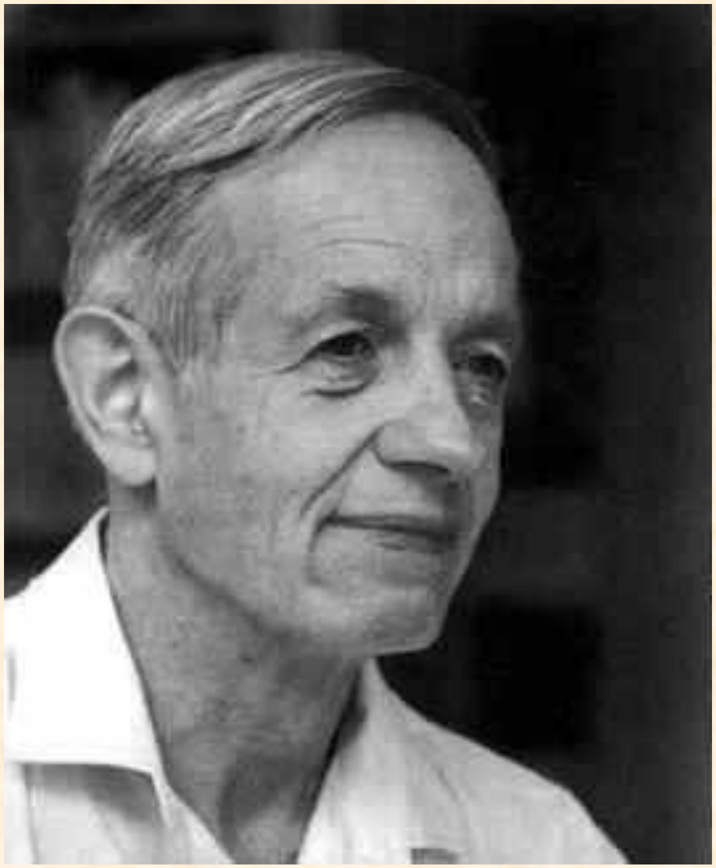
Each arc $\gamma$ corresponds to a map $\varphi: S=R/I \rightarrow k[[t]]$.

The point $(\varphi(x_1) \bmod (t), \ldots, \varphi(x_n) \bmod (t))$ satisfies each equation in I, so it lies on Spec(S). This is the image of the arc $\gamma$ under the map *pr*.

# Applications of Jet Spaces

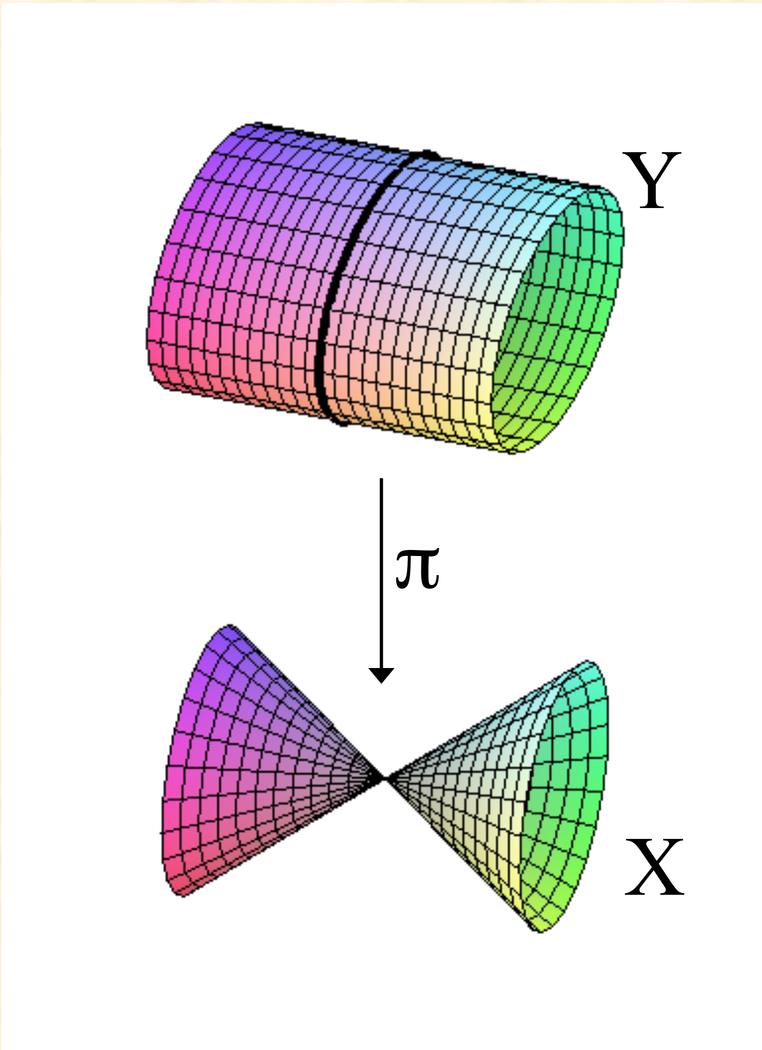
- Characterization of singularities
  - Multiplier Ideals
  - Nash Conjecture
- Motivic Integration

# Nash's Conjecture



Nash conjectured a relation between the jet space of an algebraic variety X and its resolution of singularities.

# Minimal Resolution




X: surface with isolated singularity at the origin

Y: minimal model for X (blowup and normalize)

$$\begin{tikzcd} Z \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & X & \end{tikzcd}$$

# The Nash Map (1)

Each arc centered over 0 gives a map Spec(k[[t]]) → X. The closed point goes to 0, but the generic point lifts to Y. The VCP ensures that we can complete this map to a map of schemes.

In fact, each arc gets sent into a unique exceptional divisor.

$$\begin{tikzcd} & Y \arrow[d, "\pi"'] \\ pr^{-1}(0) \subseteq J(X) \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

# The Nash Map (2)

In fact, each component of the fiber of arcs through 0 gets sent to a unique exceptional divisor, giving rise to an injective map of sets
{components of arc space through 0} → {exceptional divisors appearing in minimal resolution of singularities}

This map is known as the Nash map. Nash conjectured that this map is a bijection.

# Recent Work

The Nash conjecture has motivated research in resolution of singularities (esp. in prime characteristic) for some time [Spivakovsky, Lejeune-Jalabert].

The conjecture is true for toric varieties, surfaces and threefolds.

However, Kollar and Ishii recently gave a counterexample.

# Motivic Integration

Kontsevich developed motivic integration to prove a conjecture of Batyrev: Two birationally equivalent Calabi-Yau manifolds have the same Hodge numbers ($h_{p,q}$= dim $H^p(\Omega^q,X)$).

Get a map that sends X to $\Sigma_{p,q} h_{p,q} u^p v^q \in \mathbf{Z}[u,v]$.

Kontsevich shows this map factors through another map X → $M$ ($M$ is the motivic ring [BP]).

# The Motivic Ring

The motivic ring $M$ consists of **Z**-linear combinations of varieties plus some formal inverses. Sums correspond to disjoint unions and products correspond to direct products.

Kontsevich views the map X → $M$ as an integration on the arc space of X. He gets a change of variables formula that he uses to show that the integrals of X and Y (birational CY manifolds) are equal. Then their Hodge numbers are equal too.

# Applications: Motivic Integration

- zeta functions
- p-adic integration
- string theory
- mirror symmetry
- multiplier ideals (tight closure): singularity theory
  - Ein, Lazarsfeld and Mustata

**Thank you once again.**

**Enjoy your lunch!**