Differential Operators on Rings of Invariants

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Outline

(1) **Rings of invariants**

(a) geometric meaning and examples

(b) structural properties

(c) computational results: Derksen's algorithm

(2) **Differential operators on rings of invariants**

(a) invariant operators

(b) $D(R^G) \neq D(R)^G$

(c) computing using Gröbner bases and ideas from commutative algebra

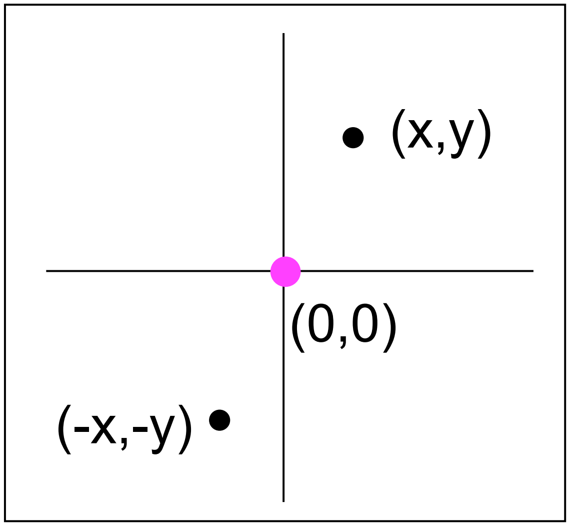
Quotient Varieties

Suppose a group G acts on a variety X.

Question: Do the orbits of G form a variety?
That is, can we make X/G into a variety?

Answer: Sometimes. When G is finite then X/G is a variety, but if G is not finite we may need to exclude certain orbits to make the set of orbits into a variety.

Example



$G = Z_2$ acting on $X = C^2$
$(x, y) \rightarrow (-x, -y)$.

The orbit space X/G is a variety.

R = C[x,y] and G acts on R
$(g \bullet r)(x,y) = r(g^{-1} \bullet (x,y))$.

$C[X/G] = \boxed{\mathbf{R^G} = \{r \in R : g \bullet r = r \text{ for all } g \in G\}}$

$= C[x^2, xy, y^2]$

$X//G = \text{Spec}(R^G)$ is the **categorical quotient**.

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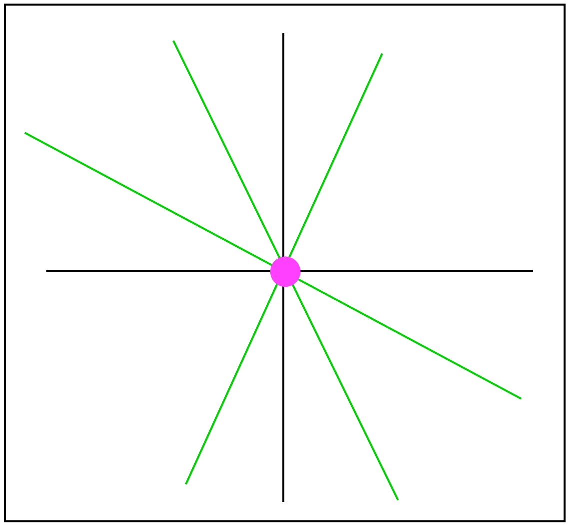
Example



$G = C^*$ acting on $X = C^2$

$t(x,y) \rightarrow (tx, ty)$

The orbit $\{(0,0)\}$ is in the closure of all other orbits.

Now X/G is not a variety.

$R^G = C$ so $X//G = pt.$

So $X//G \neq X/G$.

But $X-\{(0,0)\}/G$ is a projective variety.

In general **P**(semistable pts) is a projective variety that maps to **P**(R^G).

Summary of Examples

The categorical quotient **X//G** has coordinate ring **R^G** and in good cases it represents the orbits of semistable points in X.

This construction leads to many interesting quotient spaces:

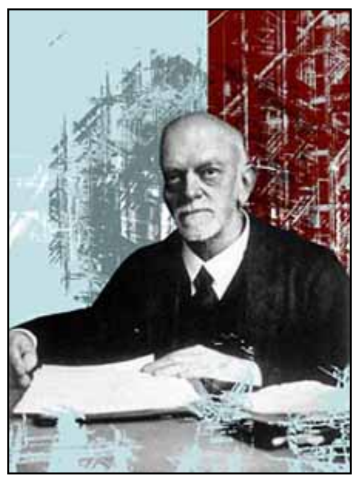
- orbifolds – G = finite group
- projective space – G = C^*
- Grassmanians G(k,n) – G = SL_nC
- moduli spaces of points (e.g. Hilbert schemes)

Properties of R^G

It is generally difficult to compute R^G explicitly.

Gordan (1868) showed that $C[X]^{SL_2C}$ is finitely generated. In 1890 Hilbert showed that R^G is finitely generated when G is linearly reductive. Nagata showed that this can fail when G is not l.r.

P. Gordan



D. Hilbert

M. Nagata

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Just use the Reynolds operator!

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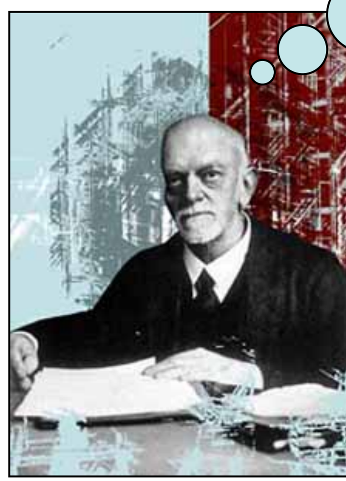

D. Hilbert

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The Reynolds operator

An algebraic group G is linearly reductive if for every G-invariant subspace W of a G-vector space V, the complement of W is G-invariant too:

$$\boxed{V = W \oplus W^c.}$$

$$\begin{tikzcd} R^G \arrow[r, shift right, "i"'] & R \arrow[l, shift right, "\mathbf{R}"'] \end{tikzcd}$$

The Reynolds operator

An algebraic group G is linearly reductive if for every G-invariant subspace W of a G-vector space V, the complement of W is G-invariant too:

$$\boxed{V = W \oplus W^C.}$$

$R^G \rightarrow R$ is a graded map and for each degree we can decompose $R_d = (R^G)_d \oplus T_d$.
As a result, we can split the inclusion by projecting onto the R^G factors.

The Reynolds operator cont.

When G is a finite group, the Reynolds operator is just an averaging operator

$$\boxed{(\mathbf{R} f)(x) = \frac{1}{|G|}\sum_{g\in G}(gf)(x)}$$

If G is infinite, then can define the Reynolds operator by integrating over a compact subgroup.

There are also explicit algebraic algorithms to compute the Reynolds operator in the case of SL_2C (see Derksen and Kemper's book).

Hilbert's wonderful proof

Thm (Hilbert): If G is lin. reductive then R^G is f.g.

Proof: Take $I = (f : f \in R^G_{>0})R$. This Hilbert ideal is f.g. because R is Noetherian. Let $f_1,\dots,f_t$ be homogeneous invs generating I. We claim that $R^G = C[f_1,\dots,f_t]$. If g in R^G_d then $g \in I$ so $g = \sum h_i f_i$ where $\deg h_i = d - \deg f_i < \deg g$.

Now $$g = \mathbf{R}(g) = \sum \mathbf{R}(h_i) f_i.$$

By induction $\mathbf{R}(h_i) \in C[f_1,\dots,f_t]$; thus so is g.

The Hochster-Roberts theorem

Thm (Hochster and Roberts): If G is linearly reductive, then R^G is Cohen-Macaulay.

An elegant proof of the result uses reduction to prime characteristic and the theory of tight closure.

Mel Hochster

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Computing invariants

Several methods:

(1) Gordan's symbolic calculus (P. Olver)

(2) Cayley's omega process

(3) Lie algebra methods (Sturmfels)

(4) Derksen's algorithm (Derksen and Kemper)

Harm Derksen and Gregor Kemper.

Derksen's Algorithm

(1) Hilbert ideal I = ideal of R gen by $R^G_{>0}$

(2) To find I, we first look at the map

$$\psi : G \times X \to X \times X \qquad B = \overline{\mathrm{im}(\psi)}$$

$$(g, x) \mapsto (x, g \bullet x) \qquad \beta = \mathrm{ideal}(B)$$

Hilbert - Mumford Criterion :

$$B \cap (X \times \{0\}) = V(I) \times \{0\}$$

$$\beta + (y_1, \ldots, y_n) = I + (y_1, \ldots, y_n)$$

Gröbner bases!

Compute the ideal β by elimination and set y's to 0 to get gens for the Hilbert ideal IR.

(3) The gens of I may not be invariants but we can apply the Reynolds operator to get invariants that generate I and R^G.

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Easy example

Let R=C[x,y,z] and G=Z_2.
Let G act on R by σ(x) = -x, σ(y)=z, σ(z) = y.

We represent G as **V**(t^2-1) and the action

$$\rho(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \rho(-1) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

by the interpolation matrix

$$\rho(t) = \begin{bmatrix} t & 0 & 0 \\ 0 & \frac{t+1}{2} & \frac{1-t}{2} \\ 0 & \frac{1-t}{2} & \frac{t+1}{2} \end{bmatrix}$$

Easy example continued

$$\rho(t) = \begin{bmatrix} t & 0 & 0 \\ 0 & \dfrac{t+1}{2} & \dfrac{1-t}{2} \\ 0 & \dfrac{1-t}{2} & \dfrac{t+1}{2} \end{bmatrix}$$

$t\bullet x = tx$

$t\bullet y = (t+1)y/2 + (1-t)z/2$

$t\bullet z = (1-t)y/2 + (1+t)z/2$

The ideal defining the graph of $\psi : \mathrm{G} \times \mathrm{X} \to \mathrm{X} \times \mathrm{X}$ is

$\boldsymbol{\beta} = (t^2 - 1,\ y_1 - t\bullet x,\ y_2 - t\bullet y,\ y_3 - t\bullet z) \subset \mathbb{C}[t,x,y,z,y_1, y_2, y_3]$

We compute the elimination ideal $\boldsymbol{\beta} \cap \mathbb{C}[x,y,z, y_1, y_2, y_3]$ and set $y_1 = y_2 = y_3 = 0$ to get $(y+z, z^2, xz, x^2)$.

Applying the Reynolds operator $\mathbf{R}(f)=[f(x,y,z)+f(-x,z,y)]/2$ gives gens for $\mathbf{R}^G$: $y+z$, y^2+z^2, $xz-xy$, and x^2.

Group actions on the Weyl algebra

Idea: replace $R=C[\mathbf{x}]$ with $D(R)=C\langle\mathbf{x},\partial_{\mathbf{x}}\rangle$ and compute **$D(R)^G$** $= C\langle\mathbf{x},\partial_{\mathbf{x}}\rangle^G$.

Extending the group action: G acts on an operator $\theta\in D(R)$ by

$$\boxed{(g\bullet\theta)(f) = g \bullet (\theta(g^{-1} \bullet f)).}$$

Concretely, if g acts on $x_1,\ldots,x_n$ by the matrix **A** then g acts on $\partial_1,\ldots,\partial_n$ by the matrix **$(A^T)^{-1}$**.

This action preserves the **defining relations** on the Weyl algebra: $\left[\partial_i, x_j\right] = \partial_i x_j - x_j \partial_i = \delta_{ij}$

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The associated graded ring GrD(R)

D(R) is filtered: $F_0 \subset F_1 \subset \ldots \subset D(R)$ F_i = ops of order $\leq i$

$GrD(R) = \bigoplus_{i \geq 0} F_i / F_{i-1}$ and $\sigma: D(R) \rightarrow GrD(R)$

$\theta \in F_i \rightarrow \theta \bmod F_{i-1}$

$$\left[\partial_i, x_j\right] = \partial_i x_j - x_j \partial_i = \delta_{ij}$$

$$\Downarrow$$

$$\left[\sigma(\partial_i), \sigma(x_j)\right] = \sigma(\partial_i)\sigma(x_j) - \sigma(x_j)\sigma(\partial_i) = 0$$

Write ξ_i for $\sigma(\partial_i)$, so that $GrD(R) = C[x_1,\ldots,x_n,\xi_1,\ldots,\xi_n]$
G acts on ξ_i just as on ∂_i.

The associated graded ring GrD(R)

D(R) is filtered: $F_0 \subset F_1 \subset \ldots \subset D(R)$ F_i = ops of order $\leq i$

$\mathrm{GrD}(R) = \bigoplus_{i \geq 0} F_i / F_{i-1}$ and $\sigma: D(R) \rightarrow \mathrm{GrD}(R)$

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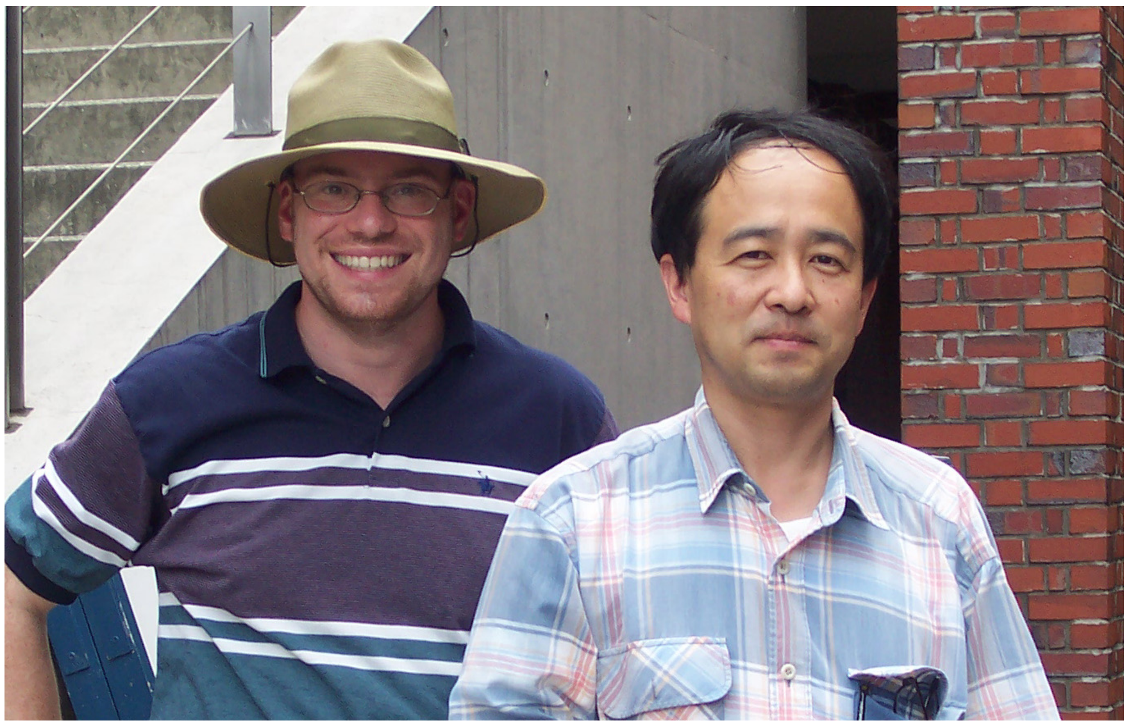
Will Traves



Nobuki Takayama

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Derksen’s algorithm applied to the Weyl algebra

Lemma: $[\mathrm{GrD}(R)]^G = \mathrm{Gr}[D(R)^G]$.

So we can apply Derksen’s algorithm to compute generators for $[\mathrm{GrD}(R)]^G = \mathrm{Gr}[D(R)^G]$.

Thm: If a set of elements generate Gr(S) then any lifts of these elements generate S.

So lifting the generators of $\mathrm{Gr}[D(R)^G]$ to $D(R)^G$ gives generators of $D(R)^G$.

Example

$R = C[x,y,z]$, $G = Z_2 = \mathbf{V}(t^2-1)$, $t\bullet x=-x$, $t\bullet y=z$, $t\bullet z=y$.

$\mathbf{R}^G = C[y+z,\ y^2+z^2,\ xz-xy,\ x^2]$.

G acts on GrD(R): $t\bullet\xi_1 = -\xi_1$, $t\bullet\xi_2=\xi_3$, $t\bullet\xi_3=\xi_2$.

$\boldsymbol{\beta} = (t^2-1,\ y_1-t\bullet x,\ y_2-t\bullet y,\ y_3-t\bullet z,\ y_4-t\bullet\xi_1,\ y_5-t\bullet\xi_2,\ y_6-t\bullet\xi_6)$

Eliminate and set y_i's to zero:

$(y+z, x^2, xz, z^2,\quad \xi_2+\xi_3, \xi_1^2, \xi_1\xi_3, \xi_3^2,\quad z\xi_3, x\xi_3, z\xi_1, x\xi_1)$.

Apply Reynolds op: $\mathbf{R}(\theta)=[\theta+\theta(-x,z,y,-\xi_1,\xi_3,\xi_2)]/2$:

$y+z$,	x^2,	y^2+z^2,	$xz-xy$,
$\xi_2+\xi_3$,	ξ_1^2,	$\xi_2^2+\xi_3^2$,	$\xi_1\xi_3-\xi_1\xi_2$,
$y\xi_2+z\xi_3$,	$x\xi_3-x\xi_2$,	$z\xi_1-y\xi_1$,	$x\xi_1$.

Lifts generate $D(R)^G$

Example

$R = C[x,y,z]$, $G = Z_2 = \mathbf{V}(t^2-1)$, $t\bullet x=-x$, $t\bullet y=z$, $t\bullet z=y$.
$\mathbf{R}^G = C[y+z,\ y^2+z^2,\ xz-xy,\ x^2]$.

G acts on GrD(R): $t\bullet\xi_1 = -\xi_1$, $t\bullet\xi_2=\xi_3$, $t\bullet\xi_3=\xi_2$.
$\boldsymbol{\beta} = (t^2-1,\ y_1-t\bullet x,\ y_2-t\bullet y,\ y_3-t\bullet z,\ y_4-t\bullet\xi_1,\ y_5-t\bullet\xi_2,\ y_6-t\bullet\xi_6)$

Eliminate and set y_i's to zero:
$(y+z, x^2, xz, z^2,\quad \xi_2+\xi_3, \xi_1^2, \xi_1\xi_3, \xi_3^2,\quad z\xi_3, x\xi_3, z\xi_1, x\xi_1)$.

Apply Reynolds op: $\mathbf{R}(\theta)=[\theta+\theta(-x,z,y,-\xi_1,\xi_3,\xi_2)]/2$:

$$\boxed{\begin{array}{llll} y+z, & x^2, & y^2+z^2, & xz-xy, \\ \xi_2+\xi_3, & \xi_1^2, & \xi_2^2+\xi_3^2, & \xi_1\xi_3-\xi_1\xi_2, \end{array}}$$
$$\begin{array}{llll} y\xi_2+z\xi_3, & x\xi_3-x\xi_2, & z\xi_1-y\xi_1, & x\xi_1. \end{array}$$

Robertz & Plesken

Computing Relations in $GrD(R)^G$ and $D(R)^G$

We can compute **relations** among the generators in $GrD(R)^G$ by elimination and then lift to relations in $D(R)^G$.

Thm (_): The lifted relations, together with the commutator relations among the generators, generate the two-sided ideal of relations in $D(R)^G$.

In our example, there are 33 nontrivial commutator relations and 40 lifted relations among the generators.

These relations are invariant under the Fourier transform $x_i \longleftrightarrow \partial_i$ because $A \in O(3)$ so $(A^T)^{-1} = A$.

The ring $D(R^G)$

Grothendieck defined a ring of differential operators $D_k(S)$ for each k-algebra S. However, $D_C(\mathbf{R^G}) \neq \mathbf{D(R)^G}$.

If $\mathbf{R^G} = C[t_1,\ldots,t_m]/J$ then

$$D_C(R^G) = \frac{\{\theta \in D(R) : \ \theta J \subseteq J\}}{\{JD(R)\}}$$

The two rings are related: **π: X → X//G** induces the inclusion $\mathbf{R^G} \to R$ and this induces a map

π_*: $D(R)^G \to D(R^G)$.

Thm (Kantor, Levasseur): When G is a finite group, π_* is **injective**.

When π_* fails to be surjective

Example: $G=Z_2$ acts on $R=C[x]$ by $x \rightarrow -x$. $R^G=C[x^2]$ so $D(R^G)$ is a Weyl algebra, but $D(R)^G = C<x^2, x\partial, \partial^2>$ so π_* is not surjective.

The group G is generated by **pseudoreflections** ($g\in G$ a **pseudoref** iff $\rho(g)$ has eigenvalues $1,1,\ldots,1,\eta$.)

Thm (Sheppard-Todd-Chevalley): R^G is a poly ring and $D(R^G)$ is a Weyl algebra if and only if G is generated by pseudoreflections.

No pseudoreflections ➔ π_* surjective

Thm (Kantor, Levasseur): When G is a finite group the map π_* is surjective precisely when G contains **no pseudoreflections**. In such cases, **$D(R)^G = D(R^G)$.**

In our running example, G contains no pseudorefs and so $D(R^G) = D(R)^G$.

When G contains some pseudoreflections, they generate a normal subgroup $P \triangleleft G$. Then

$$\boxed{R^G = (R^P)^{G/P} \text{ and } D(R^G) = \pi_*(D(R^P)^{G/P}),}$$

where now R^P is a poly ring, $D(R^P)$ is a Weyl algebra, and the map π_* comes from the inclusion $R^G \subset R^P$.

Example

$$G = \langle \tau, \gamma, \delta : \sigma^2 = \tau^2 = \delta^2 = 1, \gamma\delta = \tau\sigma, \tau\delta = \delta\tau \rangle \qquad |G| = 8.$$

$$\rho(\gamma) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \ \rho(\tau) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \rho(\delta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

$P = \langle \tau, \delta \rangle$ is the subgroup generated by pseudoreflections.

If $R = C[x, y, z]$ then $R^P = C[x, y^2, z^2]$.

Relabel these X, Y and Z.

$G/P \cong Z_2$ acts on R^P as in our running example, so $D(R^G)$ is just as in our running example, only the invariants are in X, Y and Z.

Grassmanians

The Grassmanian $\mathbf{G}(k-1,n-1)$ of (k-1)-planes in $\mathbf{P}^{n-1}$ can be realized as a quotient variety.

Each subspace represented by a choice of basis:

$$\begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ v_1 & v_2 & \cdots & v_k \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

SL_nC acts on C^n in a natural way and acts on a the matrix M by change of basis: $g \bullet M_x = M_{[g \bullet x]}$.

semistable points = full rank matrices
and the quotient identifies those whose columns generate the same subspace of C^n.

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The variety **G**(2,4)

If we assign coordinates to the n by k matrix and compute SL_nC invariants, $R^G=C[x_{jk}]^{SL_2C}$ then we obtain the coordinate ring of the cone over the Grassmanian. This is the variety **G**(k,n) of k-planes in C^n.

We describe **G**(2,4) and D(**G**(2,4)). Write V for C^2 and compute **G**(2,4) = Spec($C[V^4]^{SL_2C}$).

Fundamental Theorem of Invariant Theory:
The ring R^G is generated by the determinants of the 2x2 minors of our matrix, $[jk] = x_{j1}x_{k2} – x_{k1}x_{j2}$, subject to the **Plücker relation**: $[12][34] – [13][24] + [14][23] = 0$

The ring $GrD(R)^G$

The **FTIT** applies more generally, to $\mathbb{C}[V^4 \oplus V^{*4}]^{SL_2\mathbb{C}}$

This turns out to be precisely **$GrD(R)^G$**. We have:

6 Plücker coordinates,

6 similar coordinates in the ξ_i's and

16 mixed coordinates involving both x_i's and ξ_j's.

As well, **Popov and Vinberg** described the relations among these invariants.

$[jk] = x_{j1}x_{k2} - x_{k1}x_{j2}$ = det of j^{th} and k^{th} columns

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix}$$

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$|j\ k| = \xi_{j1}\,\xi_{k2} - \xi_{k1}\,\xi_{j2}$ = det of j^{th} and k^{th} columns

$$\begin{bmatrix} \xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} \\ \xi_{21} & \xi_{22} & \xi_{23} & \xi_{24} \end{bmatrix}$$

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$\langle j\ k\rangle = x_{j1}\,\xi_{k2} + x_{j1}\,\xi_{k2}$ = dot product of j^{th} and k^{th} cols

The ring $GrD(R)^G$

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As well, **Popov and Vinberg** described the relations among these invariants.

Lifting these generators gives generators for $D(R)^G$.

Thm(_): The relations can also be lifted to $D(R)^G$.

What about the ring $D(\mathbf{R}^G)$?

The ring $D(\mathbf{R}^G)$

What about the ring $D(\mathbf{R}^G)$? Need to use the map π_*.

Schwarz showed that π_* is surjective in this case and has kernel generated by $D(R)g \cap D(R)^G$.

Thm(_): The kernel of this map is a principal 2-sided ideal and is generated by the Casimir operator, an operator generating the center of g.

Gerry Schwarz

$C = \theta(\theta+2) - 4\ \Sigma\ [jk]\ |jk|$ where

$\theta = \langle 11\rangle + \langle 22\rangle + \langle 33\rangle + \langle 44\rangle$

Thank you.
Danke schön.