Invariants and Differential Operators

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Agenda

- Invariant Theory
  - Group actions
  - Rings of invariants
    - Reynolds operator

- New invariants from old
  - Differential conditions
  - The symmetry algebra and $D(R^G)$
    - Computing $D(R^G)$
    - Properties of $D(R^G)$

Group Actions

When a group G acts on a set X we can consider the orbit space X/G.

We'll focus on the case where we have a linear representation of G: $X \cong k^n$.

Example: If $G = \langle \sigma : \sigma^2 = e \rangle$ acts on the line $X = R$ by $\sigma(x) = -x$ then X/G is a half-line.

Examples of G-actions

Example: $G = \mathbf{C}^*$ acts on $X = \mathbf{C}^2 \setminus \{(0,0)\}$ by dilation $t(x,y) = (tx, ty)$.

The orbits are the punctured lines through the origin and X/G is just the projective space $\mathbf{P}^1_{\mathbf{C}}$.

Unhappy G-actions

Modify the last example a little: let $G = \mathbb{C}^*$ act on $X = \mathbb{C}^2$ via dilation $t(x,y) = (tx, ty)$.

The orbits are: • the origin itself

- **punctured lines through the origin**

X/G is not an algebraic variety!

Therapy for unhappy G-actions

- **Surgery: Remove the offending orbits from the original space**
  - **work only with the semistable orbits**
  - **used in G.I.T. to construct moduli spaces**

- **Less invasive: work with an algebraic version of the orbit space, the categorical quotient X//G.**

Invariants

Invariants are used to define $X//G$.
The action of G on X induces an action of G on functions $f\colon X \to C$ via $(gf)(x) = f(g^{-1}x)$.

The function f is invariant if $gf = f$ for all $g \in G$.

It is a relative invariant if $gf = \chi(g)f$ for some character $\chi\colon G \to C^*$.

The ring of invariants

The (relative) invariants form a subring R^G of $R = C[X]$.

The categorical quotient $X//G$ is just $\mathrm{Spec}(R^G)$.

Example: categorical quotient

If $G = \langle \sigma : \sigma^2 = e \rangle$ acts on the plane C^2 by

$$\sigma(x,y) = (-x,-y)$$

then the orbit space C^2 / G is a surface

$C[x,y]^G = C[x^2, xy, y^2]$ (polys of even degree)

Here all orbits are semistable and
$C^2 / G = C^2 // G = \mathrm{Spec}\ C[x^2, xy, y^2]$

In general, $X // G = X / G$ when all the orbits of G have the same dimension.

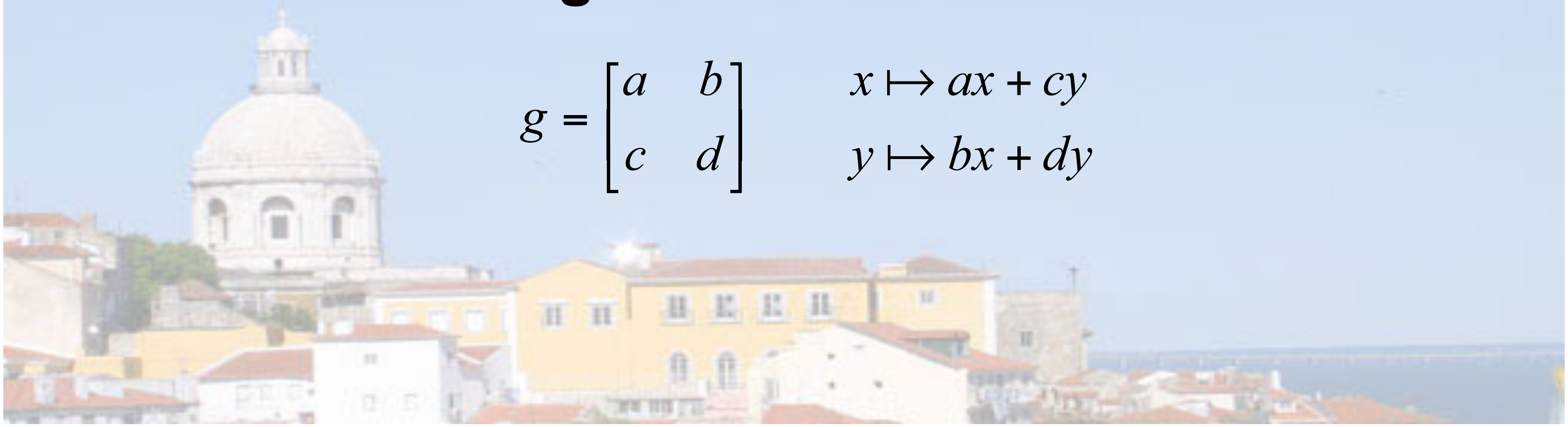
Example: binary forms

The space $S_d(C^2)$ of degree d forms is

$$\left\{ a_d x^d + a_{d-1}\binom{d}{d-1} x^{d-1} y + \cdots + a_1 \binom{d}{1} x y^{d-1} + a_0 y^d : a_i \in C \right\}$$

If g $\in$ G=Gl_2C acts on C^2 then g acts on C[x,y] via the matrix g

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{aligned} x &\mapsto ax + cy \\ y &\mapsto bx + dy \end{aligned}$$



Binary forms continued

When we plug into the form

$$a_d x^d + a_{d-1}\binom{d}{d-1}x^{d-1}y + \cdots + a_1\binom{d}{1}xy^{d-1} + a_0 y^d$$

our coefficients change. So we get an induced action on the coefficients (this is the rep. $Sym^2(C^{d+1})$)

Let $R=C[a_0,a_1,\ldots,a_d]$ and let R^G be the ring of relative invariants

$$\boxed{R^G = \{ f \in R\text{: for some } w \text{ and all } g\in G,\ gf = (\det g)^w f \}}$$

Binary forms continued

There is a correspondence between Gl_2C invariants of weight w and homogeneous Sl_2C invariants of degree 2w/d. Sl_2C invariants of binary forms encode information about the geometry of points on the projective line.

Example: $C[a_0,a_1,a_2]^{Sl_2C} = C[a_0a_2 - a_1^2]$

Example: $C[a_0,a_1,a_2,a_3,a_4]^{Sl_2C} = C[S,T]$

$S = a_0a_4 - 4a_1a_3 + 3a_2^2$ **and**

$T = a_0a_2a_4 - a_0a_3^2 + 2a_1a_2a_3 - a_1^2a_4 - a_3^2$

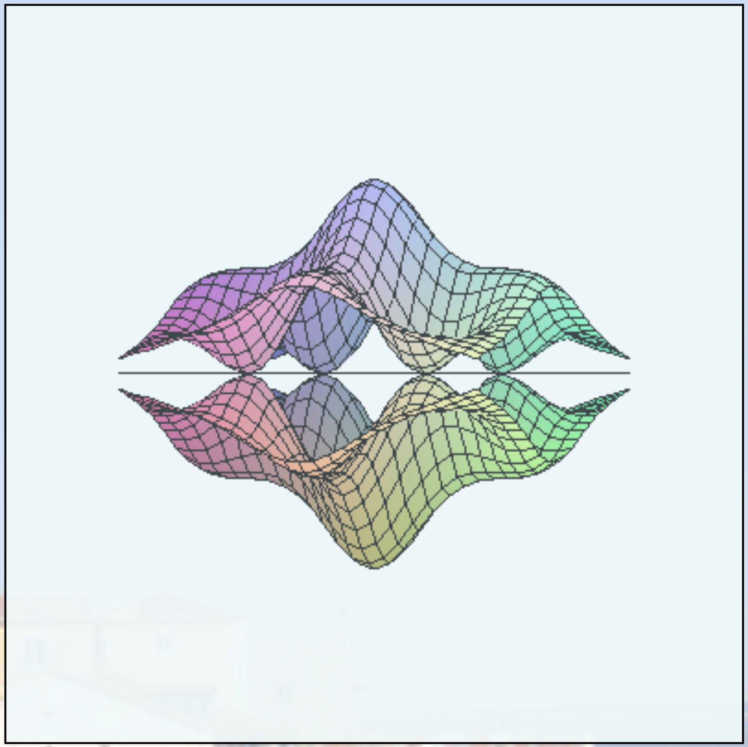
Elliptic Curves

This last example has to do with elliptic curves.

Every elliptic curve is a double cover of P^1, branched at 4 points.

$j(E)$ = j-invariant of E

$$= S^3/(S^3 - 27T^2)$$



Finite Generation

All the rings of invariants we've seen so far have been finitely generated. Gordan proved that $C[X]^{Sl_2C}$ is finitely generated (1868) but his methods don't extend to other groups.

Paul Gordan

King of the invariants

Ring of the invariants

$$R^G$$

Hilbert's Finiteness Theorem

Just use the
Reynolds operator!

David Hilbert

In 1890 Hilbert shocked the mathematical community by announcing that rings of invariants for linearly reductive groups are always finitely generated.

The Reynolds operator

An algebraic group G is linearly reductive if for every G-invariant subspace W of a G-vector space V, the complement of W is G-invariant too:
$V = W \oplus W^C$.

$$\begin{tikzcd} R^G \arrow[r, shift right, "i"'] & R \arrow[l, shift right, "\mathbf{R}"'] \end{tikzcd}$$

The Reynolds operator

An algebraic group G is linearly reductive if for every G-invariant subspace W of a G-vector space V, the complement of W is G-invariant too:
$V = W \oplus W^C$.

$R^G \rightarrow R$ is a graded map and for each degree we can decompose $R_d = (R^G)_d \oplus T_d$.
As a result, we can split the inclusion by projecting onto the R^G factors.

The Reynolds operator cont.

When G is a finite group, the Reynolds operator is just an averaging operator

$$(\mathbf{R}f)(x) = \frac{1}{|G|}\sum_{g\in G}(gf)(x)$$

If G is infinite, then can define the Reynolds operator by integrating over a compact subgroup.

There are also explicit algebraic algorithms to compute the Reynolds operator in the case of Sl_2C (see Derksen and Kemper's book).

Hilbert's wonderful proof

Thm (Hilbert): If G is lin. reductive then R^G is f.g.

Proof: Take $I = (f : f \in R^G_{>0})R$. This Hilbert ideal is f.g. because R is Noetherian. Let $f_1,\ldots,f_t$ be homogeneous generators of I. We claim that $R^G = C[f_1,\ldots,f_t]$. If g in R^G_d then $g \in I$ so $g = \sum h_i f_i$ where $\deg h_i = d - \deg f_i < \deg g$.

Now $$g = \mathbf{R}(g) = \sum \mathbf{R}(h_i) f_i.$$

By induction $\mathbf{R}(h_i) \in C[f_1,\ldots,f_t]$; thus so is g.

The Hochster-Roberts theorem

Thm (Hochster and Roberts): If G is linearly reductive, then R^G is Cohen-Macaulay.

An elegant proof of the result uses reduction to prime characteristic and the theory of tight closure.

Mel Hochster

Computing invariants

Several methods:
(1) Gordan’s symbolic calculus (P. Olver)
(2) Cayley’s omega process
(3) Grobner basis methods (Sturmfels)
(4) Derksen’s algorithm (Derksen and Kemper)

Harm Derksen

Gregor Kemper

Derksen's Algorithm

(1) It is enough to find generators of the Hilbert ideal I = $R^G_{>0}$.
(2) These may not generate R^G but their images under the Reynolds operator will.
(3) To find I, we first look at the map

$$\psi : G \times V \to V \times V \qquad \mathrm{B} = \overline{\mathrm{im}(\psi)}$$

$$(\mathrm{g}, \mathrm{v}) \mapsto (\mathrm{v}, \mathrm{gv}) \qquad \mathrm{b} = \mathrm{ideal}(\mathrm{B})$$

Hilbert - Mumford Criterion :

$$\mathrm{B} \cap (\mathrm{V} \times \{0\}) = \mathrm{V}(I) \times \{0\}$$

$$\mathrm{b} + (\mathrm{y}_1, \ldots, \mathrm{y}_\mathrm{n}) = I + (\mathrm{y}_1, \ldots, \mathrm{y}_\mathrm{n})$$

Compute the ideal b by elimination and set y's to zero to get generators for the Hilbert ideal.

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New invariants from old

Question (2001): Is the HS-algebra the same as the Steenrod algebra?

Larry Smith

Steenrod algebra $\subset$ Weyl algebra in prime characteristic

S.A. acts on both C[X] and $C[X]^G$ and so it can be used to produce new invariants from old.

Turns out: SA $\neq$ HS (INGO 2003).
But the question got me thinking about diff. ops. and rings of invariants.

Symmetry algebra for $H_A(\beta)$

Together with M. Saito: Studied the symmetry algebra for any hypergeo. system $H_A(\beta)$:

$$A \in Z_{d\times n}, \quad \beta \in C^d$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad \begin{aligned} (\theta_1 + \theta_2 + \theta_3) f &= \beta_1 \\ (\quad \theta_2 + 2\theta_3) f &= \beta_2 \\ (\partial_1\partial_3 - \partial_2^2) f &= 0 \end{aligned}$$

The solutions to these systems are connected to a toric variety and if $\theta \in S_A$ then $\theta(f)$ is a solution to a new hypergeo. system $H_A(\beta')$.

Differential conditions

Question: Can we use differential operators to produce new invariants from known invariants?

Relation between differential equations and invariants (due to Cayley; see Hilbert, 1897).

Arthur Cayley

We'll develop these conditions for the Sl_2C invariants of the binary forms but the basic idea is that the invariants form a module over the Weyl algebra.

Torus Invariants

Have a torus T^2 sitting in Gl_2C as the diagonal and the invariants f under T^2 must satisfy

$$\mathbf{f(\lambda x,\tau y)=(\lambda\tau)^w f(x,y).}$$

$$A = \begin{bmatrix} 0 & 1 & \cdots & d \\ d & d-1 & \cdots & 0 \end{bmatrix}$$

$$\begin{aligned} a_d &\mapsto \lambda^d \tau^0 a_d \\ a_{d\text{-}1} &\mapsto \lambda^{d-1}\tau^1 a_{d\text{-}1} \\ &\vdots \end{aligned}$$

$$f(x,y) = a_0^{k_0} a_1^{k_1} \cdots a_d^{k_d} + \cdots$$

$$f(\lambda x, \tau y) = \lambda^{dk_d + (d-1)k_{d\text{-}1} + \cdots + k_1} \tau^{dk_0 + (d\text{-}1)k_1 + \cdots + k_{d\text{-}1}} a_0^{k_0} a_1^{k_1} \cdots a_d^{k_d} + \cdots$$

$$= \lambda^w \tau^w a_0^{k_0} a_1^{k_1} \cdots a_d^{k_d} + \cdots$$

$$\Rightarrow \begin{cases} dk_0 + (d-1)k_1 + \cdots + k_{d-1} = w \\ k_1 + 2k_2 + \cdots + dk_d = w \end{cases}$$

Torus Invariants

Have a torus T^2 sitting in Gl_2C as the diagonal and the invariants f under T^2 must satisfy

$$\mathbf{f(\lambda x,\tau y)=(\lambda\tau)^w f(x,y).}$$

$$\sum i a_i \partial_i f = wf$$

$$\sum (d - i) a_i \partial_i f = wf$$

$$\Rightarrow \begin{cases} \sum i a_i \partial_i f = wf \\ d \sum a_i \partial_i f = d \deg f = 2wf \end{cases}$$

The other two generators

Along with the torus, Gl_2C is generated by two other kinds of matrices

$$\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix}$$

$$\boxed{\sum i a_{i-1} \partial_i f = 0} \qquad \boxed{\sum (d - i) a_{i+1} \partial_i f = 0}$$

The DE for binary forms

$$\begin{cases} (\sum i a_i \partial_i - w) f = 0 \\ (\sum a_i \partial_i - 2w/d) f = 0 \\ (\sum i a_{i-1} \partial_i) f = 0 \\ (\sum (d-i) a_{i+1} \partial_i) f = 0 \end{cases}$$

homogeneity

isobaric

if and only if $f(a_0,a_1,\ldots,a_d)$ is a relative invariant of weight w and degree 2w/d.

Questions on invariant DEs

$$\begin{cases} \boxed{\begin{array}{c} (\sum i a_i \partial_i - w) f = 0 \\ (\sum a_i \partial_i - 2w/d) f = 0 \end{array}} \\ \boxed{\begin{array}{c} (\sum i a_{i-1} \partial_i) f = 0 \\ (\sum (d-i) a_{i+1} \partial_i) f = 0 \end{array}} \end{cases}$$

Question: When is this system holonomic?
In these cases, find a formula (or bounds) for the holonomic rank of this system.
(Hint: Molien series gives a lower bound)

Symmetry algebra

Given a left ideal J in the Weyl algebra D(R), the symmetry algebra of J is

$$S\left(\frac{D(R)}{J}\right) := \frac{\{\theta \in D(R) : J\theta \subset J\}}{J}$$

Consider the left ideal J in D(R) generated by the Cayley's system of differential equations. What does its symmetry algebra look like?

Symmetry for Cayley's system

$$\boxed{\mathrm{S}\left(\frac{\mathrm{D(R)}}{\mathrm{J}}\right) := \frac{\{\theta \in \mathrm{D(R)} : \mathrm{J}\theta \subset \mathrm{J}\}}{\mathrm{J}}}$$

$$\mathrm{f} \in \mathrm{R}^{\mathrm{G}} \Leftrightarrow J \bullet f = 0$$

$$\text{Now } \mathrm{J} \bullet (\theta \bullet \mathrm{f}) = (\mathrm{J}\theta) \bullet \mathrm{f} \subset \mathrm{J} \bullet \mathrm{f} = 0$$

$$\text{so } \theta \bullet \mathrm{f} \in \mathrm{R}^{\mathrm{G}} \text{ if } \mathrm{f} \in \mathrm{R}^{\mathrm{G}} \text{ and } \theta \in \mathrm{S(D(R)/J)}.$$

$$\boxed{\mathrm{S}\left(\frac{\mathrm{D(R)}}{\mathrm{J}}\right) \cong \frac{\{\theta \in \mathrm{D(R)} : \theta \bullet \mathrm{R}^{\mathrm{G}} \subset R^G\}}{\{\theta \in \mathrm{D(R)} : \ \theta \bullet \mathrm{R}^{\mathrm{G}} = 0\}} \subset D(R^G)}$$

New invariants from old

Recall our question: can we use operators to produce new invariants from old?

The naïve answer is Yes! Just use operators in $D(R^G)$. But this is often badly behaved.

So we'll try to use its subring $S(D(R)/J)$ instead.

Questions: How do we compute $S(D(R)/J)$ and what algebraic properties does it have?

When is R^G a simple module over $S(D(R)/J)$?

Invariant operators

If G acts on R then it also acts on the Weyl algebra D(R): if $g \cdot x = Ax$ then $g \cdot \partial = (A^{-1})^T \partial$.

The action preserves the order filtration so it descends to the associated graded ring:

$$[\mathrm{gr}D(R)]^G = \mathrm{gr}(D(R)^G).$$

Since grD(R) is a polynomial ring, its ring of invariants is f.g. (and so is $D(R)^G$).

Unfortunately, this is not the ring $D(R^G)$.

Distinction: $D(R)^G$ versus $D(R^G)$

The map $R^G \rightarrow R$ induces a map $\pi^*: D(R)^G \rightarrow D(R^G)$.

We just get $\pi^*\theta$ by restriction. Or we can view the map as:

$$\begin{tikzcd}
R \arrow[r, "\theta"] & R \arrow[d, "\mathcal{R}"] \\
R^G \arrow[u, "i"] \arrow[r, dashed, "\pi^*\theta"'] & R^G
\end{tikzcd}$$

Theorem: $\mathrm{Im}(\pi^*) \subset S(D(R)/J) \subset D(R^G)$.

Failure of surjectivity

We've got a map $\pi^*: D(R)^G \rightarrow D(R^G)$.

Have $\mathrm{Im}(\pi^*) \subset S(D(R)/J) \subset D(R^G)$.

Musson and Van den Bergh showed that the map π^* may not be surjective (G = torus).

Ian Musson

M. Van den Bergh

Surjective when it counts

Schwarz showed that the map π^* is surjective in many cases of interest. In fact, he showed that the Levasseur-Stafford Alternative holds for Sl_2C representations:

Either (1) R^G is regular or
(2) the map π^* is surjective at the graded level

Gerry Schwarz

Computing $D(R^G)$

In most cases the ring R^G is not regular and we have

$$\mathrm{im}\ \pi^* = S(D(R)/J) = D(R^G).$$

In these cases, we have the analogue of my result with M. Saito for the $H_A(\beta)$.

We can compute generating sets for these rings by applying π^* to lifts of a generating set for $[\mathrm{gr}D(R)]^G$. In particular, for all Sl_2C representations, $D(R^G)$ is finitely generated.

Simple Results

For G lin. red, R^G is a simple module over $D(R^G)$.

$D(R^G)$ itself is often a simple ring. For instance, this is known for tori (Van den Bergh) and for many classical groups (Levasseur and Stafford).

Thm (Smith,VdB): $D(R^G)$ is simple for all lin. red. G in prime characteristic!

It remains open whether $D(R^G)$ is always simple.

Recap

- **Invariant Theory**
  - **Group actions**
  - **Rings of invariants**
    - **Reynolds operator**

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