MAAC 2004
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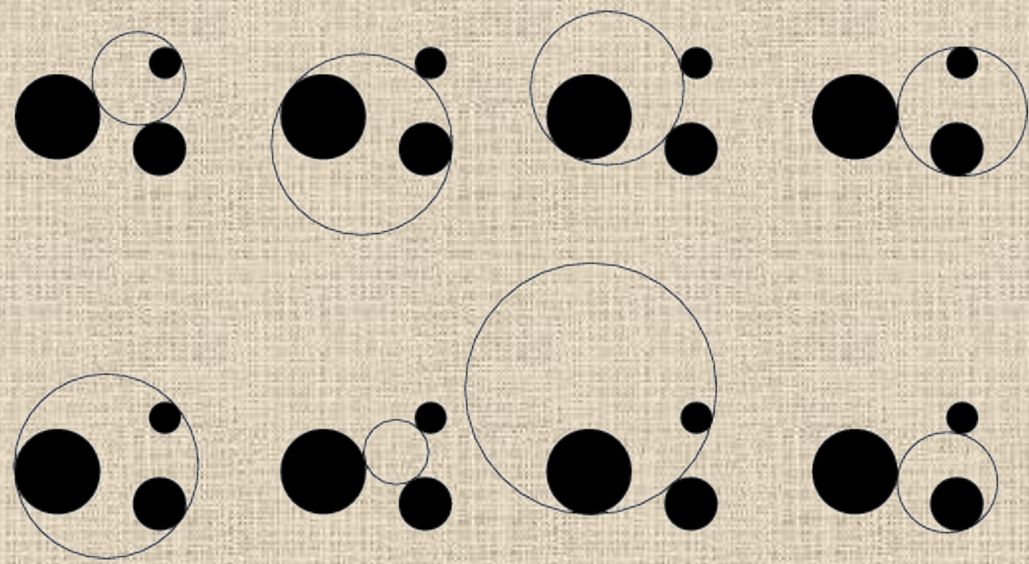

Conics on the Cubic Surface

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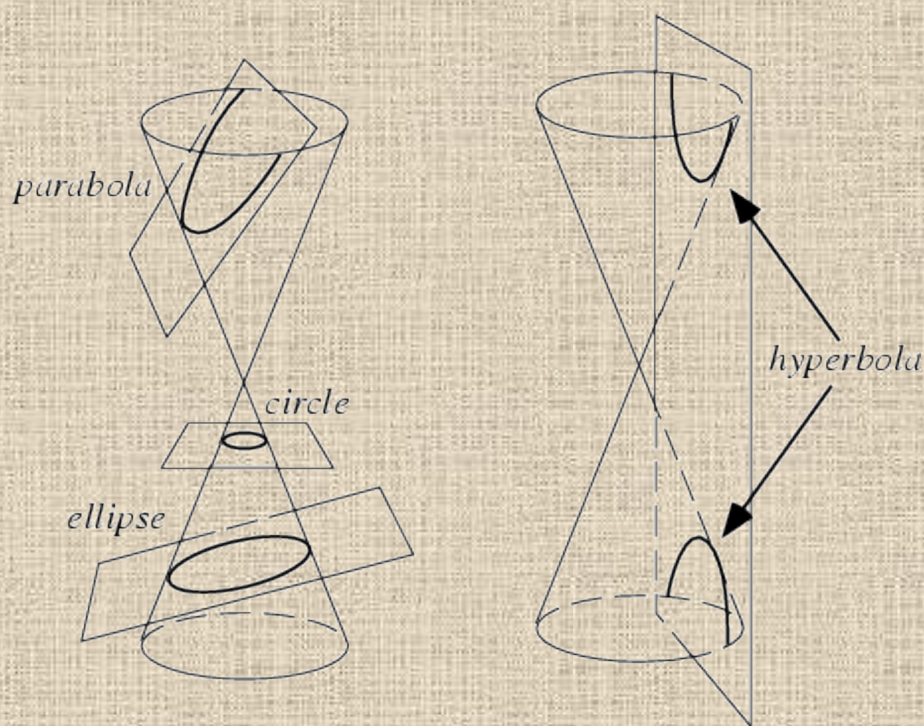
USNA Trident Project

This talk is a preliminary report on joint work with MIDN 1/c Andrew Bashelor and my colleague at USNA Amy Ksir.

Bashelor’s Trident project focuses on enumerative algebraic geometry and its connections with Quantum cohomology.

Conics in the Plane

Conics in R^2 are well-known -- circles, ellipses, hyperbolas, parabolas, and even pairs of lines and double lines are familiar geometric objects. We are interested in conics in the projective plane P_C^2. In this setting, all conics are **projectively equivalent**.



Moreover, each nonsingular conic is a **rational curve**: it can be parameterized as the image of a degree 2 map from P^1 to P^2.

For example, $x^2 + y^2 = z^2$ is parameterized by

$x = (s^2 – t^2)/2$

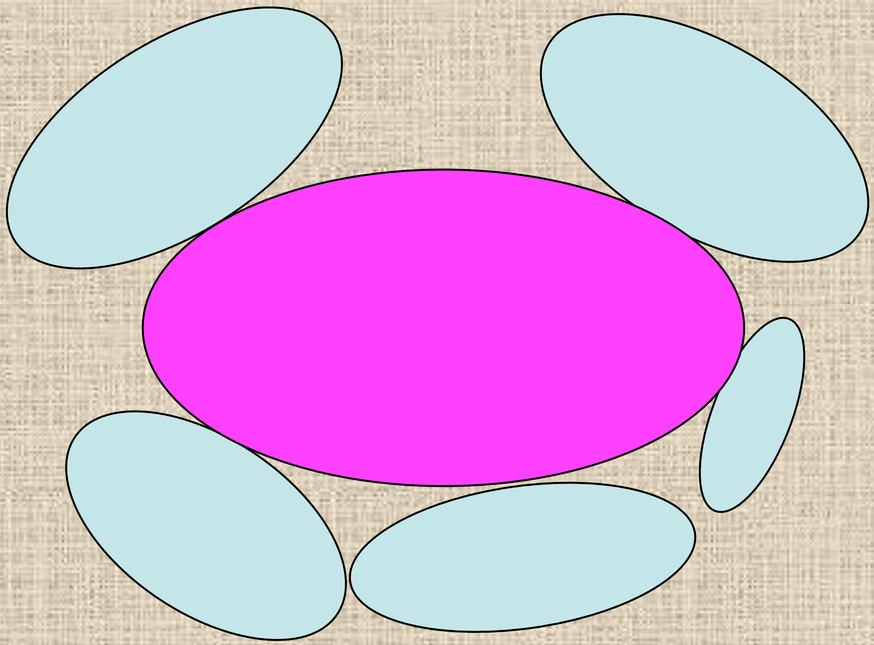
$y = st$

$z = (s^2 + t^2)/2$.

Steiner’s Problem

How many conics are tangent to five given plane conics?

Jakob Steiner 1796 - 1863



Parameterizing Conics

Each conic is given by an equation with 6 coefficients:

$$ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz = 0$$

So the space of conics is a five dimensional projective space P^5;
the conic above corresponds to the point (a: b: c: d: e: f).

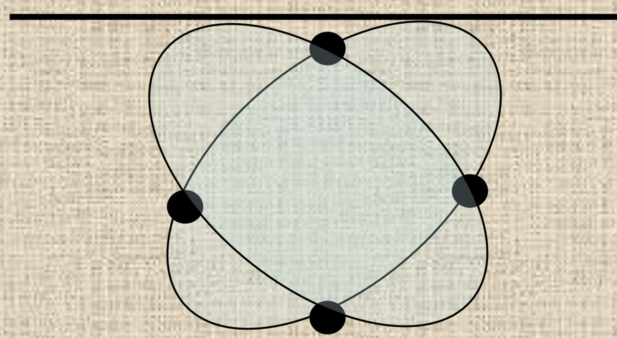
QUESTION: How many conics pass through 5 general points in P^2?

Each point imposes a homogeneous linear condition on the coefficients of the conic. When the points are in general position, these conditions are linearly independent and so there is a unique solution to these equations.

So there is precisely one conic through 5 general points.
CABRI is a geometry program that nicely illustrates this fact.

Conics Tangent to Lines

QUESTION: How many conics pass through 4 points and are tangent to a line in general position?



Each point imposes a linear condition $T_P = 0$ on the coefficients of the conic.

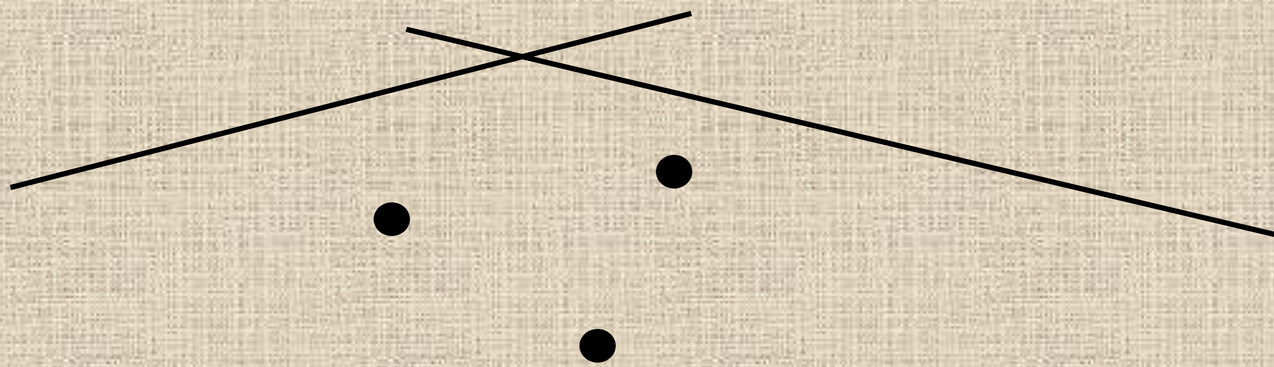
Parameterize the conic and restrict the defining equation for the line to the parameterized curve. This gives a homogeneous degree 2 polynomial in the parameters s and t. The conic is tangent to the line precisely when this polynomial has a single root; that is, when its **discriminant** (“$b^2 - 4ac$”) vanishes. This gives a degree 2 condition in the coefficients of the conic $T_L = 0$.

Bézout’s Theorem states that if we have n hypersurfaces (with no common component) of degrees $d_1, \ldots, d_n$ in P^n then their intersection consists of $d_1 d_2 \ldots d_n$ points, counted with appropriate multiplicities.

So the solution to 4 linear equations and one quadratic equation in P^5 should consist of two points. There are two conics through the 4 points and tangent to a given line.

Conics Tangent to Lines^2

QUESTION: How many conics pass through 3 points and are tangent to 2 lines in general position?

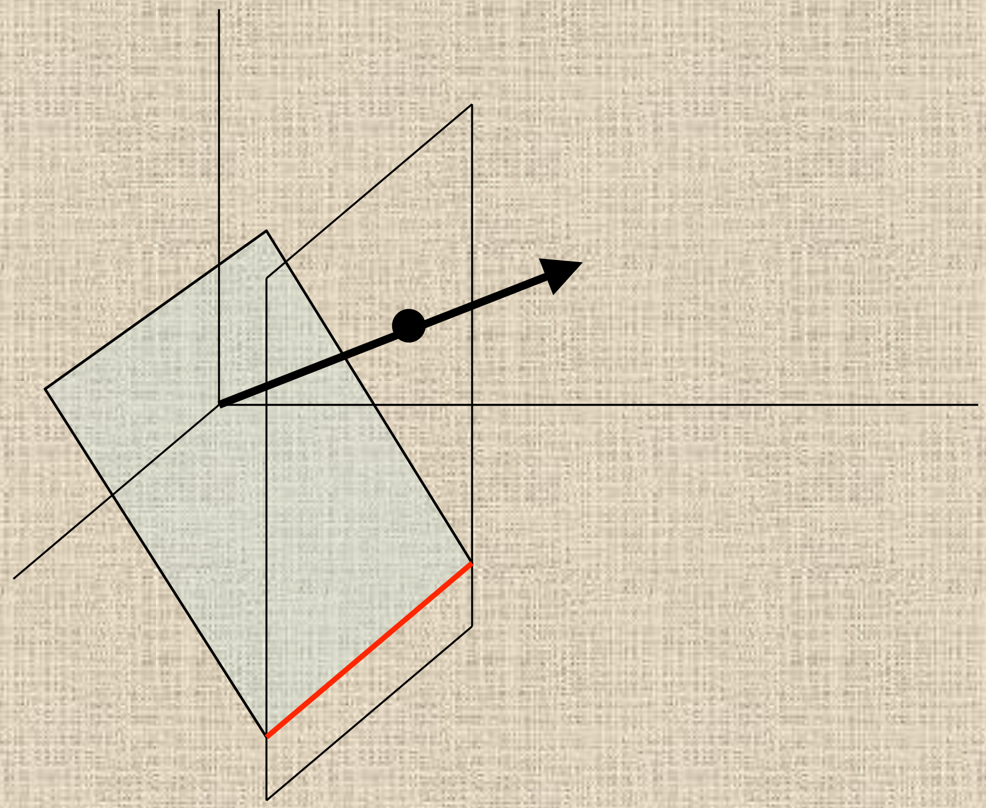


We can apply the same techniques to see that these conics should satisfy 3 linear equations and two quadratic equations in P^5. This leads to four solutions. (Depending on the position of the points and lines, some of these solutions may be complex).

More Tangent Conics

When fewer than 3 points are specified, there are double lines that pass through the points and are (automatically) tangent to every line.

This causes problems with our counting method but we can get answers by **duality**.



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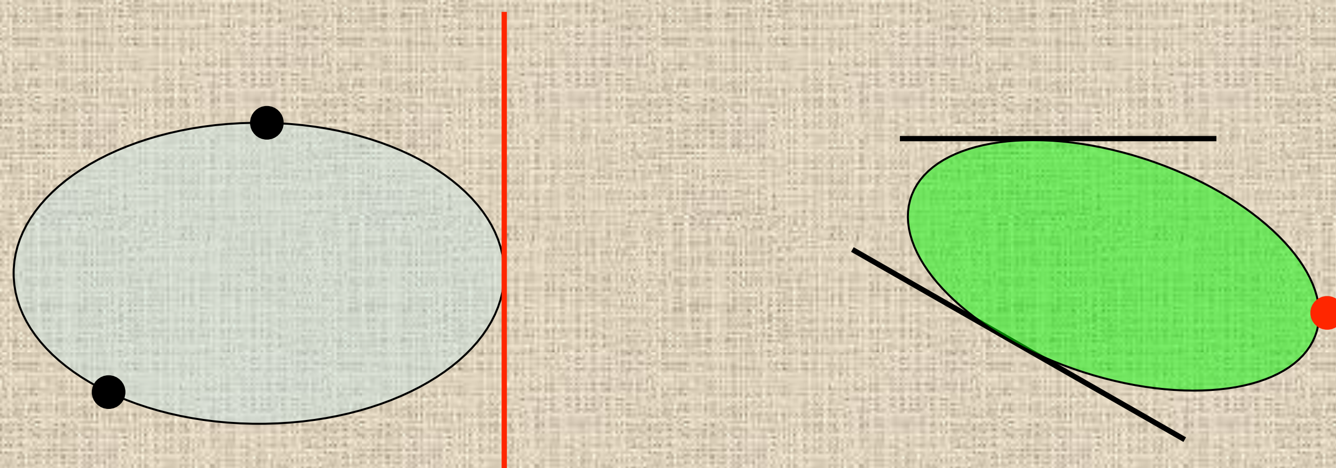
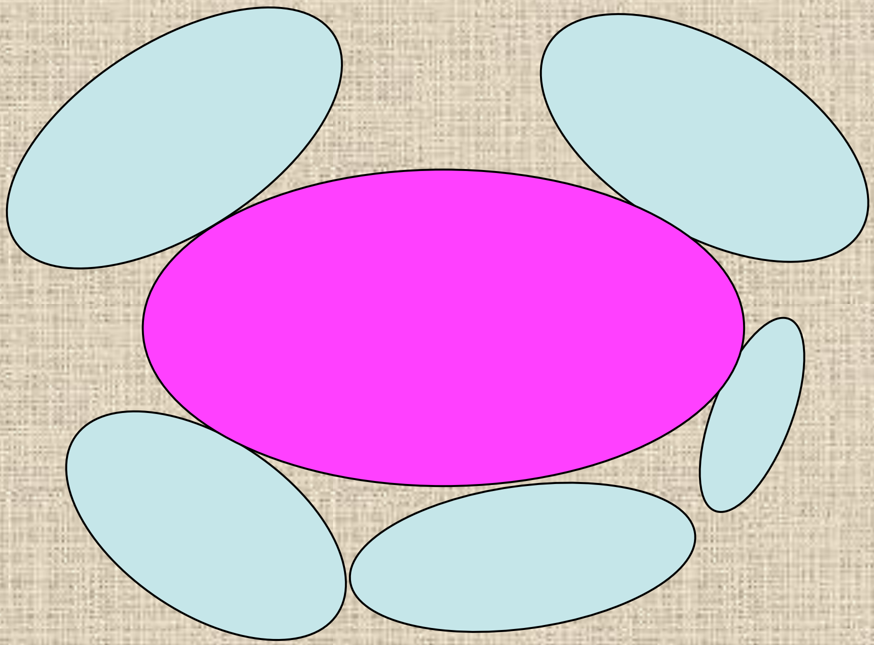


There is a symmetry in the answers to the counting questions: there is a unique conic tangent to 5 lines, two conics tangent to 4 lines and through a point and four conics tangent to 3 lines and through 2 points.

Steiner's Problem

How many conics are tangent to five given plane conics?

Jakob Steiner 1796 - 1863



Steiner's Initial Approach

Steiner followed a similar approach to that taken earlier. Each conic is given by an equation with 6 coefficients:

$$ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2f\,yz = 0$$

So the space of conics is a five dimensional projective space P^5.

Each of the fixed conics Q_i can be parameterized, realized as a degree two embedding of P^1 into P^2:

$$x = X(s,t) \quad y = Y(s,t) \quad z = Z(s,t)$$

A general conic meets a fixed conic at the points (s:t) where

$$aX^2 + bY^2 + cZ^2 + 2dXY + 2eXZ + 2f\,YZ = 0$$

and is tangent to the fixed conic when this polynomial has a multiple root; that is, when its discriminant vanishes. The discriminant of this polynomial is a degree 6 polynomial T_{Q_i} in the variables a, b, c, d, e, and f.

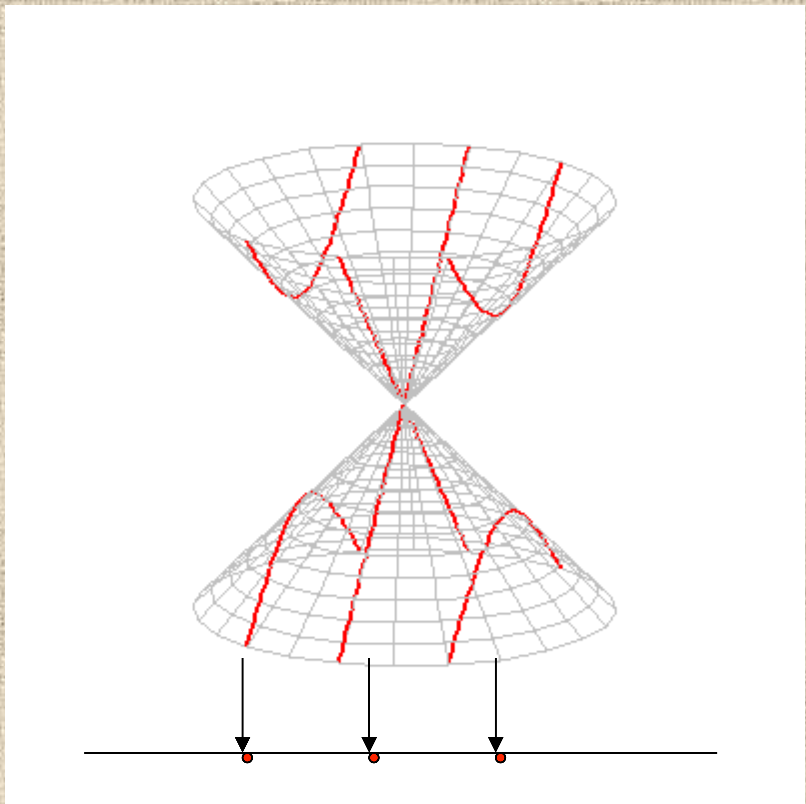
Steiner's Initial Answer

Each of the five fixed conics gives rise to a condition T_Q. These conditions constrain the coefficients of the conic we are looking for. So the number of such conics should be the number of solutions to the system

Since each $T_Q = 0$ gives a hypersurface of degree 6, Steiner initially thought that there should be $6^5 = 7776$ solutions.

Very quickly he realized that this was incorrect because each of the $T_Q = 0$ contains the Veronese surface parameterizing double lines (the 2-uple embedding of P^2 into P^5). So Bézout's Theorem does not apply!

The Chow Ring



When varieties vary in a (flat) family each variety is said to be rationally equivalent to the others.

The equivalence class of the intersection of two varieties depends only on the equivalence classes of the varieties, so we can talk about $[V] \cap [V'] = [V \cap V']$.

We define a ring where our objects are equivalence classes of varieties, addition is formal (think union of varieties) multiplication corresponds to the intersection of two varieties.

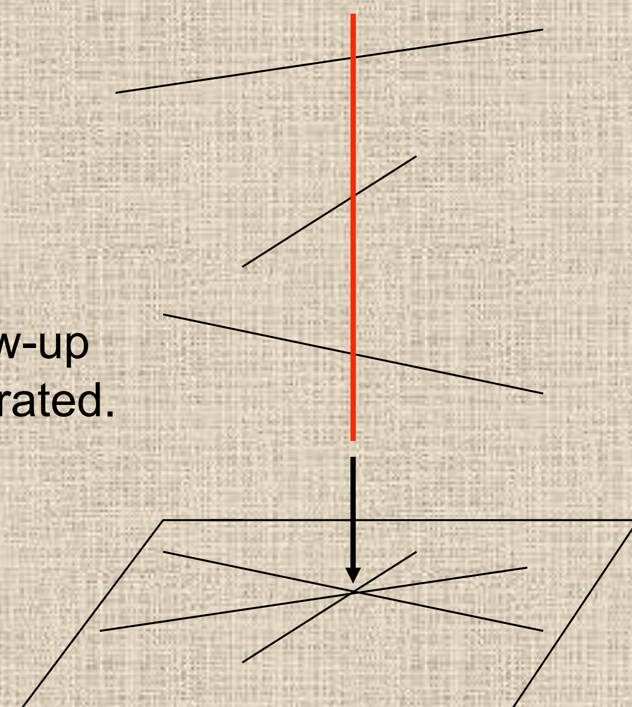
In Steiner's problem, we want to find $[\cap T_{Q_i}] = [T_Q]^5$.

Blowing up

We can find the correct solution using an idea communicated to me by Frank Sottile. Since the hypersurfaces $T_Q = 0$ intersect in the Veronese surface, we blow-up P^5 along the Veronese to get a space often called the **space of complete conics** (with blow-up map $\pi: X \rightarrow P^5$).

Frank Sottile

Blowing up causes hypersurfaces which intersected on the blow-up locus to become separated.



Computations on the Blow-up

After blowing up the total transform is $\pi^*(\mathrm{T_Q}) = \widetilde{\mathrm{T}}_\mathrm{Q} + 2\mathrm{E} = 6\widetilde{\mathrm{H}}$

One way to calculate the answer is to find $(\widetilde{\mathrm{T}}_\mathrm{Q})^5 = (6\widetilde{\mathrm{H}} - 2\mathrm{E})^5$

But this requires knowing the ring structure of the Chow ring for the blow-up. Another way is to use the classes T_P and T_L that we defined earlier:

$$\pi^*(\mathrm{T_P}) = \widetilde{\mathrm{T}}_\mathrm{P} = \widetilde{\mathrm{H}}$$

$$\pi^*(\mathrm{T_L}) = \widetilde{\mathrm{T}}_\mathrm{L} + \mathrm{E} = 2\widetilde{\mathrm{H}}$$

$$\Longrightarrow$$

$$\widetilde{\mathrm{H}} = \widetilde{\mathrm{T}}_\mathrm{P}$$

$$\mathrm{E} = 2\widetilde{\mathrm{T}}_\mathrm{P} - \widetilde{\mathrm{T}}_\mathrm{L}$$

$$\widetilde{\mathrm{T}}_\mathrm{Q} = 6\widetilde{\mathrm{H}} - 2\mathrm{E} = 2\widetilde{\mathrm{T}}_\mathrm{P} + 2\widetilde{\mathrm{T}}_\mathrm{L}$$

It follows that $(\widetilde{T}_Q)^5 = (2\widetilde{T}_P + 2\widetilde{T}_L)^5$

$$= 32(\widetilde{T}_P^5 + 5\widetilde{T}_P^4\widetilde{T}_L^1 + 10\widetilde{T}_P^3\widetilde{T}_L^2 + 10\widetilde{T}_P^2\widetilde{T}_L^3 + 5\widetilde{T}_P^1\widetilde{T}_L^4 + \widetilde{T}_L^5)$$

$$= 32(1 + 5(2) + 10(4) + 10(4) + 5(2) + 1)$$

$$= 3264.$$

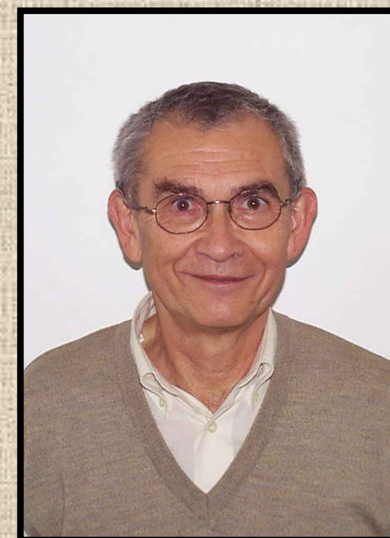
Rational Plane Curves

M. Kontsevich

Kontsevich and Manin found a beautiful
I recursion formula relating plane
curves of various degrees. If

n_d = the number of rational degree d
plane curves through 3d-1 general points

then clearly $n_1 = 1$ and $n_2 = 1$.
It turns out that $n_3 = 12$ and



Y. Manin

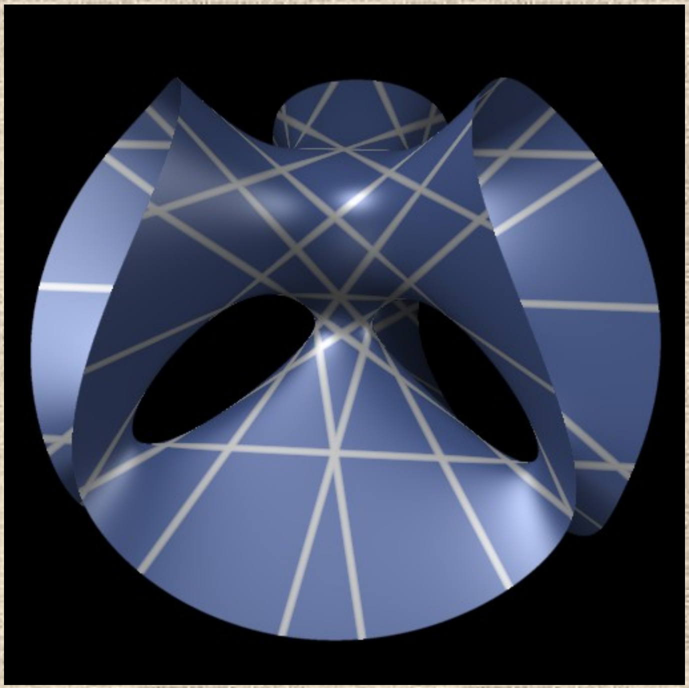
$$n_d = \sum_{d_1+d_2=d} n_{d_1} n_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right)$$

The proof uses the associativity of the quantum cohomology ring of Grassmanians in a crucial and elegant way.

The Cubic Surface

Inspired by Kontsevich and Manin, we tried to count the number of rational curves of given degree on another well-known surface, the smooth cubic surface.

This cubic surface S is the blowup of P^2 at 6 points. There is a four dimensional space of cubics passing through all 6 points and choosing a basis for this space gives a parameterization $P^2 \rightarrow P^3$ whose image is S.



It is a classical result that there are 27 lines on the cubic surface. These correspond to the images of the 15 lines through pairs of points, the 6 conics through groups of 5 points and the 6 exceptional divisors.

Clebsch and Salmon

Degree of a Curve on S

For a curve C on S, we define deg(C) to be the number of points of intersection of a general hyperplane in P^3 with C. Since the cubics through the six points embed S into P^3 the intersection of a generic hyperplane with S is the strict transform of a generic cubic through the six points.

If C is the strict transform of a curve C' on P^2 that is a degree t curve and passes through each of the points P_i with multiplicity n_i then $\deg(C) = C.\tilde{G}$ where G is a cubic on P^2.

$$
\begin{aligned}
3t &= \pi^*(G).\pi^*(C') = (\tilde{G} + E_1 + \cdots + E_6).(C + n_1E_1 + \cdots + n_6E_6) \\
&= \tilde{G}.C + \sum n_i(\tilde{G}.E_i) + \sum C.E_i + \sum n_i(E_i.E_j) \\
&= \tilde{G}.C + \sum n_i
\end{aligned}
$$

Thus,

$$\boxed{\deg(C) = 3t - \sum n_i} \qquad \text{(D)}$$

Rational Curves on S

We want to consider rational (genus 0) curves. In this case, the adjunction formula 2g-2 = C.(C+K) gives

$$
\begin{aligned}
-2 &= C.(C - 3H + E_1 + \cdots + E_6) \\
&= (tH - n_1E_1 - \cdots - n_6E_6).((t-3)H - n_1E_1 - \cdots - n_6E_6) \\
&= t(t-3) - n_1(n_1 - 1) - \cdots - n_6(n_6 - 1)
\end{aligned}
$$

Solving gives the condition

$$\boxed{\sum_{i=1}^{6} n_i(n_i - 1) = (t-2)(t-1)} \qquad \text{(A)}$$

Counting Curves on S

Solving the two equations

$$\deg(C) = 3t - \sum n_i \qquad \text{(D)}$$

$$\sum_{i=1}^{6} n_i(n_i - 1) = (t-2)(t-1) \qquad \text{(A)}$$

when deg(C) = 1 gives 27 lines. When deg(C) is 2, we get conics on S. There are 27 integer solutions to the equations (A) and (D) when deg(C) = 2 and each corresponds to a family of curves parameterized by P^1. There are 27 conics passing through a general point of S.

CONJECTURE: There are finitely many rational degree d curves through d-1 general points on S. Call this number s_d.

We have $s_1 = 27$, $s_2 = 27$, $s_3 = 72$, $s_4 = 216$, $s_5 = 459$, $s_6 = 936$, ….

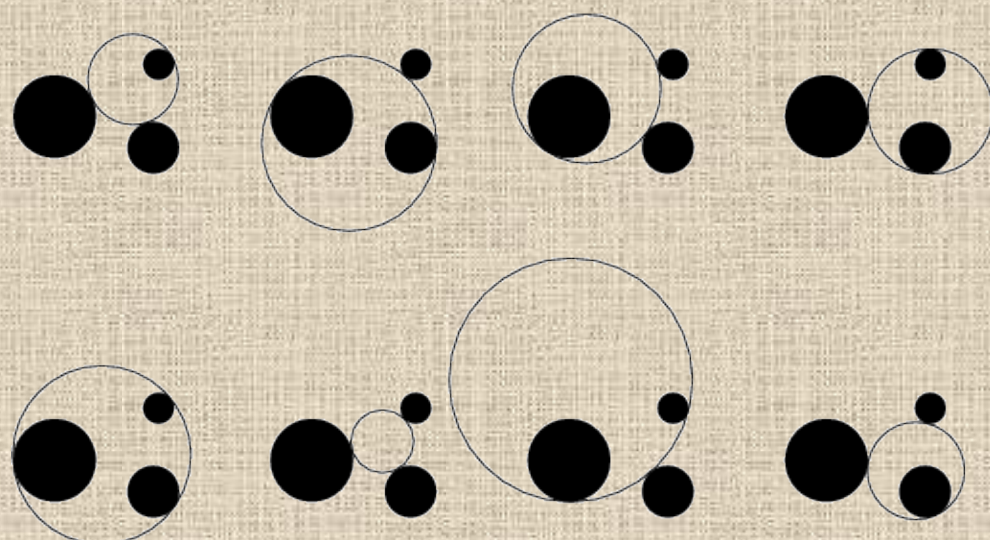
QUESTION: Is there a recursion between the s_d?

Apollonius Circles

QUESTION: Given three circles, find all circles tangent to these three.

This question was solved by Apollonius in his lost “Tangencies”.

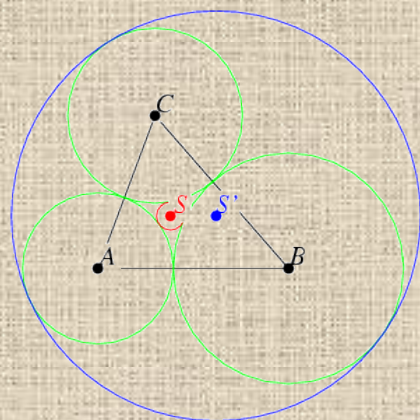
Note that every circle $(x-a)^2 + (y-b)^2 = r^2$ when homogenized must pass through the line at infinity in two special points $(1:i:0)$ and $(1:-i: 0)$. Smooth conics passing through these two points will be called **circles**. Imposing this condition reduces the space of conics to a P^3. The tangency conditions are degree 2 hypersurfaces that intersect in finitely many points. There are thus 8 solutions to the problem above.



A Very Special Case

When the three circles are mutually tangent, then there are two solutions to the problem. This was known to Descartes. Soddy found a beautiful formula relating the curvatures of the three given circles to that of a tangent circle:

$$2(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2) = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)^2$$



Our counting must be done with care here. Each of the two solutions pictured count as 1, but each of the three original circles also solve the problem! Each counts as 2 solutions.

CONJECTURE: The maximum number of mutually tangent conics in P_C^2 is 6.