Reconstructing Hyperplane Arrangements

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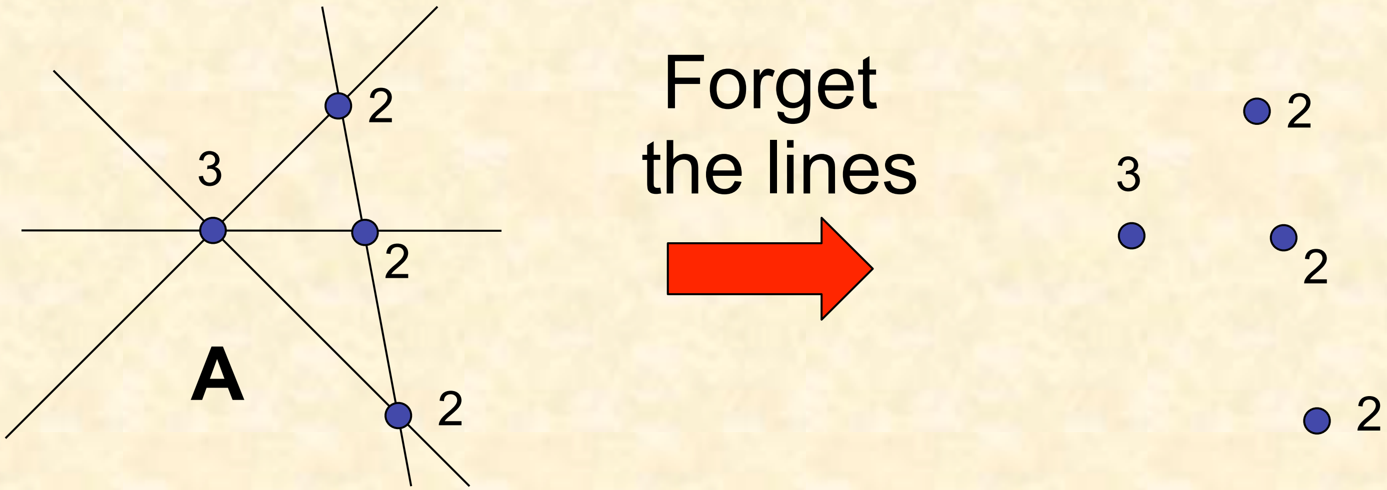
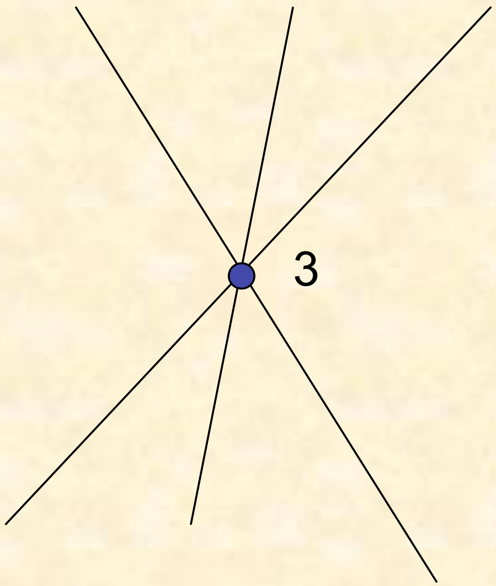
A question about lines and points

A: finite collection of lines in $\mathbf{R}^2$, no two of which are parallel



Question: Can we recover the lines from the points of intersection and their multiplicities?

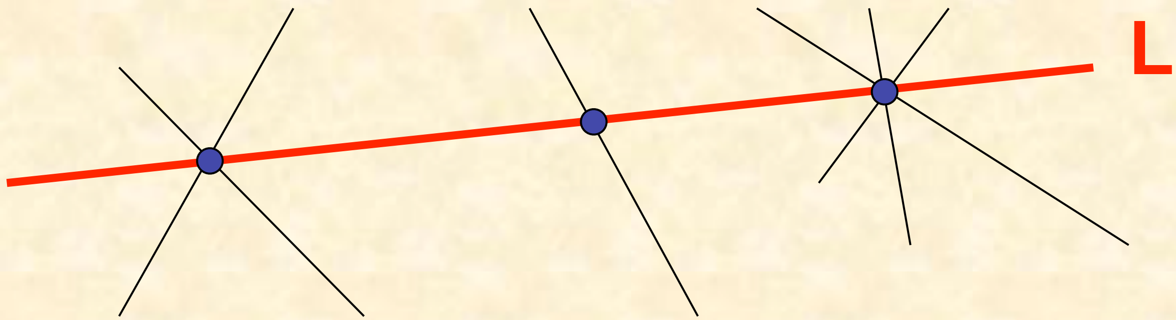
Not Possible!



No way to recover the lines if they all lie in a pencil.

Theorem (Wakefield & Yoshinaga): Can recover the lines if there is more than one intersection point.

Wakefield and Yoshinaga's idea



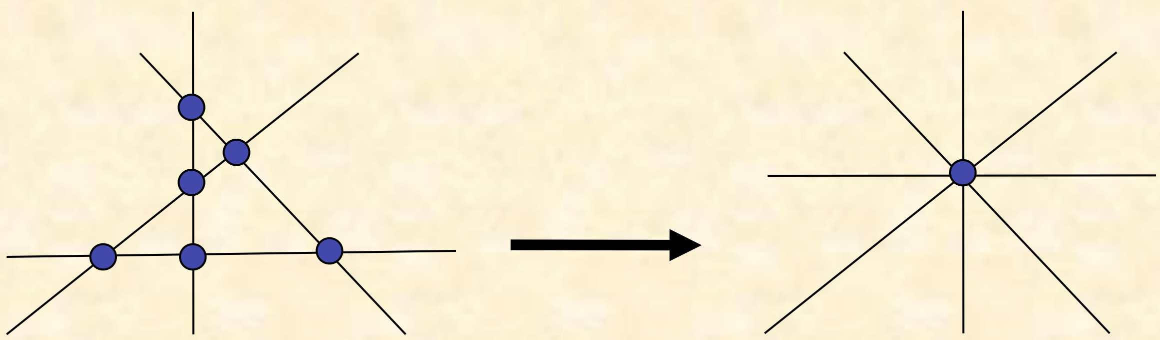
For a line **L** in **A**, $\sum_{p \in L} (d_p - 1) = n-1$.

This number is smaller for lines that are not in **A**.

An algorithm to find the lines

We can recover the **number** of lines:
If there are n lines then there ought to be $n(n-1)/2$ points of intersection.

How should we count points with multiplicity > 2?



Multiplicity d counts for $d(d-1)/2$ points.

$$\boxed{\Sigma\, d_p(d_p-1)/2 = n(n-1)/2}$$

A linear system

Let the n lines have equations $L_i = a_i x + b_i y + c_i = 0$.

Define $Q(x,y) = L_1 L_2 \ldots L_n = \sum_{s+t \leq n} k_{st}\, x^s y^t$.

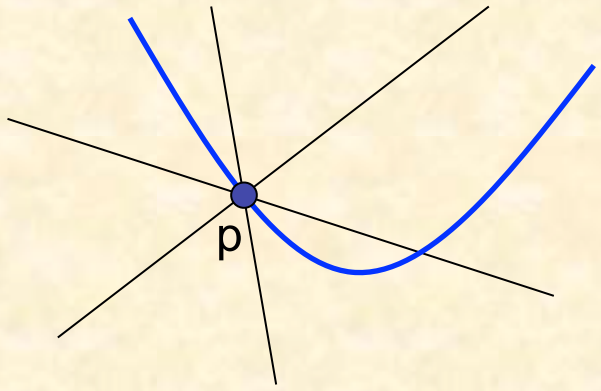
Now if p is a point of multiplicity d then Q and all its derivatives of order < d vanish at p.

These conditions give rise to **linear conditions** on the coefficients k_{st}. The system of linear equations has at least one solution, corresponding to the original collection of lines.

Bézout's Theorem

In fact, there is a unique polynomial of degree n that satisfies the multiplicity conditions.

Bézout's Theorem: Two curves of degrees m and n that do not share a common component meet in at most m·n points.



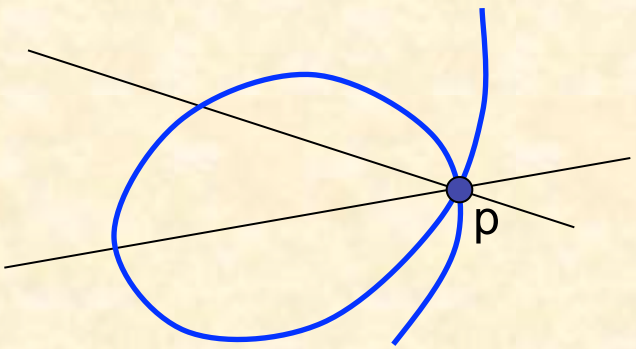
Intersection multiplicity at p = 3

Intersection multiplicity: Length $C[x,y]_p/(F,G)$ as a $C[x,y]_p$-module

transverse curves → 1

There can only be one!

There is a unique polynomial of degree n that satisfies the multiplicity conditions.



If there is another such curve, then its intersection multiplicity with the configuration of lines at the point p is at least d_p^2.

For a line L in **A**, $\sum_{p \in L} (d_p - 1) = n-1$ so $\Sigma\, d_p > n$.

But $\Sigma\, d_p(d_p-1)/2 = n(n-1)/2$ so

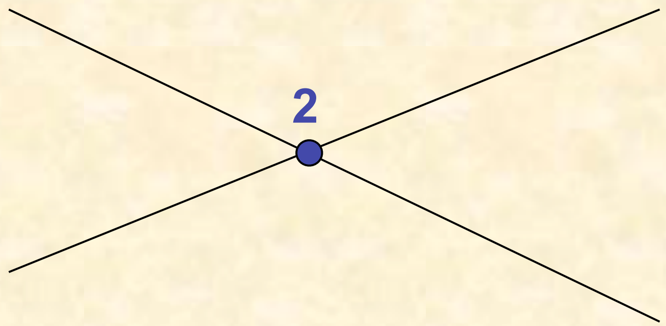
$$\Sigma\, d_p^2 = n^2 - n + \Sigma\, d_p > n^2,$$

a contradiction unless the curves share a component.

Removing the components

Remove the component from both curves and reduce all the multiplicities of the intersection points on the line by 1. Continue in this way.

Eventually we find ourselves at the situation where we need to reconstruct two lines passing through a point!



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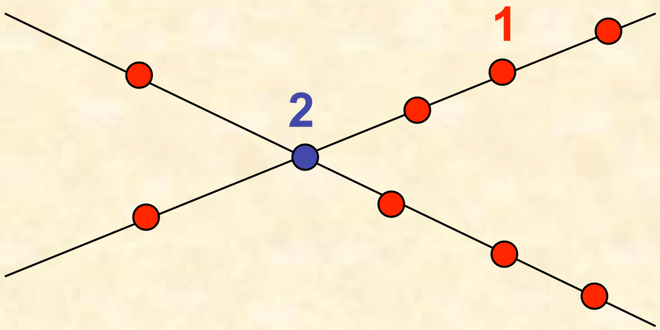


This is unique since we still have many points of multiplicity **1**.

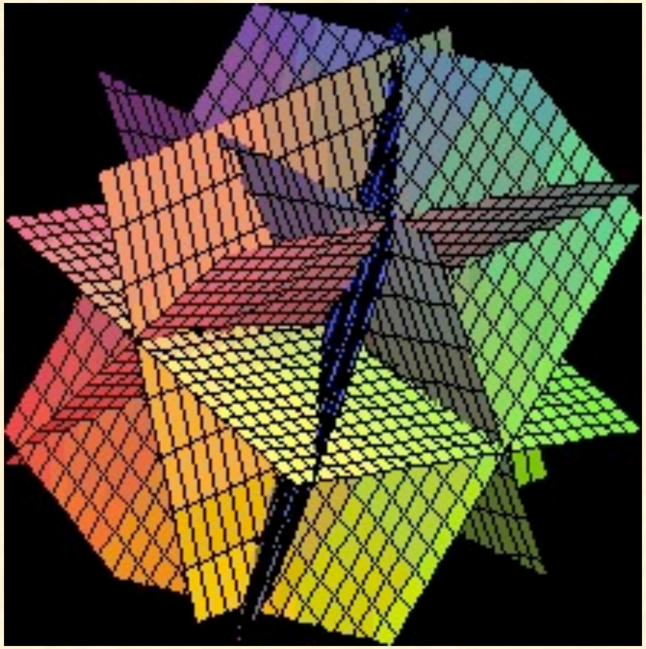
Extensions

We actually showed that the original collection of lines is the unique curve of degree n passing through the points with the given multiplicities.

And we gave an algorithm – solve an associated linear system – to find this unique polynomial of degree n.

The theorem for hyperplanes

If **A** is a collection of hyperplanes, not all sitting in a pencil, then the collection can be recovered from its intersection locus (with multiplicities).



The arrangement is the unique solution to a system of linear equations.

To see that it is unique, we use a higher-dimensional version of Bézout's theorem.